# V17 Metabolic networks - Graph connectivity

**Graph connectivity** in biological networks is related to
- finding cliques
- edge betweenness
- modular decomposition

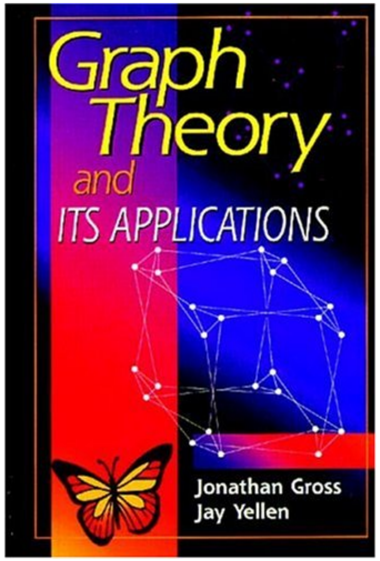
that have been covered in previous lectures.

**Cut-sets** are related to breaking up metabolic networks.

<u>Today's program</u>

V17 closely follows chapter 5.1 in the book
on the right on „Vertex- and Edge-Connectivity"

V18 will cover parts of chapter 5.3 on
„Max-Min Duality"



# Motivation: graph connectedness

Determining the number of edges (or vertices) that must be removed to disconnect a given connected graph applies directly to analyzing the **vulnerability** of existing networks.

Definition: A graph is **connected** if for every pair of vertices *u* and *v*, there is a walk from *u* to *v*.

Definition: A **component** of *G* is a maximal connected subgraph of *G*.

# Vertex- and Edge-Connectivity

Definition: A **vertex-cut** in a graph *G* is a vertex-set *U* such that *G* – *U* has more components than *G*.

A **cut-vertex** (or cutpoint) is a vertex-cut consisting of **a single vertex**.

Definition: An **edge-cut** in a graph *G* is a set of edges *D* such that *G* – *D* has more components than *G*.

A **cut-edge** (or bridge) is an edge-cut consisting of **a single edge**.

The **vertex-connectivity $\kappa_v(G)$** of a connected graph *G* is the minimum number of vertices whose removal can either disconnect *G* or reduce it to a 1-vertex graph.

→ if *G* has at least one pair of non-adjacent vertices, then $\kappa_v(G)$ is the size of a smallest vertex-cut.

# Vertex- and Edge-Connectivity

Definition: A graph $G$ is **k-connected** if $G$ is connected and $\kappa_v(G) \geq k$.
If $G$ has non-adjacent vertices, then $G$ is $k$-connected
if every vertex-cut has at least $k$ vertices.

Definition: The **edge-connectivity $\kappa_e(G)$** of a connected graph $G$
is the minimum number of edges whose removal can disconnect $G$.

$\rightarrow$ if G is a connected graph,
the edge-connectivity $\kappa_e(G)$ is the size of a smallest edge-cut.

Definition: A graph $G$ is ***k*-edge-connected**
if $G$ is connected and every edge-cut has at least $k$ edges (i.e. $\kappa_e(G) \geq k$).

# Vertex- and Edge-Connectivity

Example: In the graph below, the vertex set {*x*,*y*} is one of three different 2-element vertex-cuts. There is no cut-vertex. → $\kappa_v(G) = 2$.

The edge set {*a*,*b*,*c*} is the unique 3-element edge-cut of graph *G*, and there is no edge-cut with fewer than 3 edges. Therefore $\kappa_e(G) = 3$.

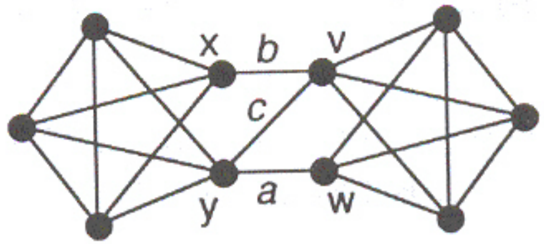



**Figure 5.1.1** A graph $G$ with $\kappa_v(G) = 2$ and $\kappa_e(G) = 3$.

Application: The connectivity measures $\kappa_v$ and $\kappa_e$ are used in a quantified model of **network survivability**, which is the capacity of a network to retain connections among its nodes after some edges or nodes are removed.

# Vertex- and Edge-Connectivity

Since neither the vertex-connectivity nor the edge-connectivity of a graph is affected by the existence or absence of self-loops, we will assume in the following that all graphs are loopless.

Proposition 5.1.1 Let *G* be a graph. Then the edge-connectivity $\kappa_e(G)$ is less than or equal to the minimum degree $\delta_{min}$ (G).

Proof: Let *v* be a vertex of graph *G* with degree $k = \delta_{min}(G)$. Then, the deletion of the *k* edges that are incident on vertex separates *v* from the other vertices of *G*. □

# Vertex- and Edge-Connectivity

Definition: A collection of distinct non-empty subsets $\{S_1, S_2, \ldots, S_l\}$ of a set A is a **partition** of A if both of the following conditions are satisfied:

(1) $S_i \cap S_j = \emptyset$ , $\forall\ 1 \leq i < j \leq l$
(2) $\cup_{i=1 \ldots l} S_i = A$

Definition: Let $G$ be a graph, and let $X_1$ and $X_2$ form a partition of $V_G$.

The set of all edges of $G$ having one endpoint in $X_1$ and the other endpoint in $X_2$ is called a **partition-cut** of G and is denoted $\langle X_1, X_2 \rangle$.

# Partition Cuts and Minimal Edge-Cuts

Proposition 4.6.3: Let $\langle X_1, X_2 \rangle$ be a partition-cut of a connected graph $G$.

If the subgraphs of $G$ induced by the vertex sets $X_1$ and $X_2$ are connected, then $\langle X_1, X_2 \rangle$ is a minimal edge-cut.

Proof: The partition-cut $\langle X_1, X_2 \rangle$ is an edge-cut of $G$, since $X_1$ and $X_2$ lie in different components of $G$ - $\langle X_1, X_2 \rangle$.

Is it minimal?

Let $S$ be a proper subset of $\langle X_1, X_2 \rangle$, and let edge $e \in \langle X_1, X_2 \rangle - S$.
By definition of $\langle X_1, X_2 \rangle$, one endpoint of $e$ is in $X_1$ and the other endpoint is in $X_2$.

Thus, if the subgraphs induced by the vertex sets $X_1$ and $X_2$ are connected, then $G – S$ is connected.

Therefore, $S$ is not an edge-cut of $G$, which implies that $\langle X_1, X_2 \rangle$ is a minimal edge-cut. □

# Partition Cuts and Minimal Edge-Cuts

Proposition 4.6.4. Let $S$ be a minimal edge-cut of a connected graph $G$, and let $X_1$ and $X_2$ be the vertex-sets of the two components of $G – S$. Then $S = \langle X_1, X_2 \rangle$.

Proof: Clearly, $S \subset \langle X_1, X_2 \rangle$, i.e. every edge $e \in S$ has one endpoint in $X_1$ and one in $X_2$. Otherwise, the two endpoints would either both belong to $X_1$ or to $X_2$. Then, $S$ would not be minimal because $S – e$ would also be an edge-cut of $G$.

On the other hand, if $e \in \langle X_1, X_2 \rangle$ - $S$, then its endpoints would lie in the same component of $G – S$, contradicting the definition of $X_1$ and $X_2$. □

Remark: This assumes that the removal of a minimal edge-cut from a connected graph creates exactly two components.

# Partition Cuts and Minimal Edge-Cuts

<u>Proposition</u> 4.6.5. A partition-cut $\langle X_1, X_2 \rangle$ in a connected graph $G$ is a minimal edge-cut of $G$ or a union of edge-disjoint minimal edge-cuts.

<u>Proof</u>: **will not be proven in lecture**

Since $\langle X_1, X_2 \rangle$ is an edge-cut of $G$, it must contain a minimal edge-cut, say $S$.

If $\langle X_1, X_2 \rangle \neq S$, then let $e \in \langle X_1, X_2 \rangle$ - $S$, where the endpoints $v_1$ and $v_2$ of $e$ lie in $X_1$ and $X_2$, respectively.

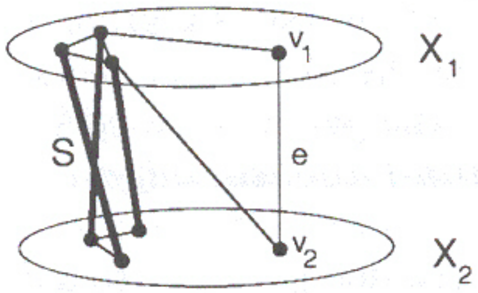



Figure 4.6.1

Since $S$ is a minimal edge-cut, the $X_1$-endpoints of $S$ are in one of the components of $G - S$, and the $X_2$-endpoints are in the other component.

Furthermore, $v_1$ and $v_2$ are in the same component of $G - S$ (since $e \in G - S$).

Suppose, wlog, that $v_1$ and $v_2$ are in the same component as the $X_1$-endpoints of $S$.

# Partition Cuts and Minimal Edge-Cuts

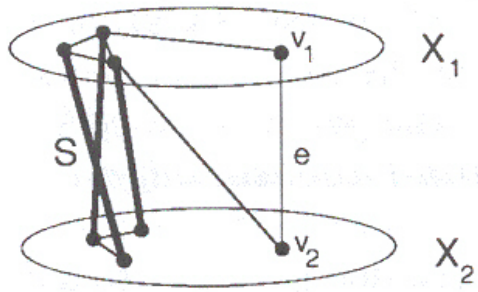



Figure 4.6.1

Then every path in $G$ from $v_1$ to $v_2$ must use at least one edge of $\langle X_1,X_2\rangle$ - $S$.

Thus, $\langle X_1,X_2\rangle$ - $S$ is an edge-cut of $G$ and contains a minimal edge-cut $R$.

Appyling the same argument, $\langle X_1,X_2\rangle$ - $(S \cup R)$ either is empty or is an edge-cut of $G$.

Eventually, the process ends with $\langle X_1,X_2\rangle$ - $(S_1 \cup S_2 \cup \ldots S_r) = \emptyset$,
where the $S_i$ are edge-disjoint minimal edge-cuts of $G$. ☐

# Partition Cuts and Minimal Edge-Cuts

Proposition 5.1.2. A graph $G$ is $k$-edge-connected
if and only if every partition-cut contains at least $k$ edges.

Proof: ($\Rightarrow$) Suppose, that graph G is $k$-edge connected.
Then every partition-cut of G has at least $k$ edges,
since a partition-cut is an edge-cut.

($\Leftarrow$) Suppose that every partition-cut contains at least $k$ edges.
By proposition 4.6.4., every minimal edge-cut is a partition-cut.
Thus, every edge-cut contains at least $k$ edges. □

# Relationship between vertex- and edge-connectivity

Proposition 5.1.3. Let $e$ be any edge of a $k$-connected graph $G$, for $k \geq 3$.
Then the edge-deletion subgraph $G - e$ is $(k - 1)$-connected.

Proof: Let $W = \{w_1, w_2, ..., w_{k-2}\}$ be any set of $k - 2$ vertices in $G - e$,
and let $x$ and $y$ be any two different vertices in $(G - e) - W$.
It suffices to show the existence of an $x$-$y$ walk in $(G - e) - W$.

First, suppose that at least one of the endpoints of edge $e$ is contained in set $W$.
Since the vertex-deletion subgraph $G - W$ is 2-connected,
there is an x-y path in $G - W$.
This path cannot contain edge e.
Hence, it is an x-y path in the subgraph $(G - e) - W$.

Next suppose that neither endpoint of edge $e$ is in set $W$.

Then there are two cases to consider.

# Relationship between vertex- and edge-connectivity

Case 1: Vertices *x* and *y* are the endpoints of edge *e*.
Graph *G* has at least $k + 1$ vertices (since *G* is *k*-connected).
So there exists some vertex $z \in G - \{w_1, w_2, ..., w_{k-2}, x, y\}$.

Since graph G is *k*-connected, there exists
an *x*-*z* path $P_1$ in the vertex deletion subgraph $G - \{w_1, w_2, ..., w_{k-2}, y\}$ and
a *z*-*y* path $P_2$ in the subgraph $G - \{w_1, w_2, ..., w_{k-2}, x\}$

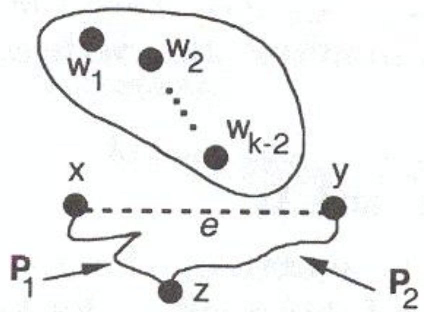



Figure 5.1.2 The existence of an $x$-$y$ walk in $(G - e) - \{w_1, w_2, \ldots, w_{k-2}\}$.

Neither of these paths contains edge *e*, and, therefore,
their concatenation is an *x*-*y* walk in the subgraph $(G - e) - \{w_1, w_2, ..., w_{k-2}\}$

# Relationship between vertex- and edge-connectivity

Case 2: At least one of the vertices $x$ and $y$, say $x$, is not an endpoint of edge $e$.
Let $u$ be an endpoint of edge $e$ that is different from vertex $x$.
Since graph $G$ is $k$-connected, the subgraph $G - \{w_1, w_2, \ldots, w_{k-2}, u\}$ is connected.

Hence, there is an $x$-$y$ path $P$ in $G - \{w_1, w_2, \ldots, w_{k-2}, u\}$.

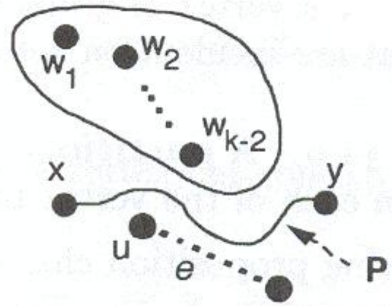



**Figure 5.1.2** The existence of an $x$-$y$ walk in $(G - e) - \{w_1, w_2, \ldots, w_{k-2}\}$.

It follows that $P$ is an $x$-$y$ path in $G - \{w_1, w_2, \ldots, w_{k-2}\}$ that does not contain vertex $u$ and, hence excludes edge $e$
(even if $P$ contains the other endpoint of $e$, which it could).

Therfore, $P$ is an $x$-$y$ path in $(G - e) - \{w_1, w_2, \ldots, w_{k-2}\}$. □

# Relationship between vertex- and edge-connectivity

Corollary 5.1.4. Let *G* be a *k*-connnected graph, and let *D* be any set of *m* edges of *G*, for $m \leq k - 1$. Then the edge-deletion subgraph $G – D$ is $(k – m)$-connected.

Proof: this follows from the iterative application of proposition 5.1.3. □

Corollary 5.1.5. Let *G* be a connected graph. Then $\kappa_e(G) \geq \kappa_v(G)$.

Proof. Let $k = \kappa_v(G)$, and let *S* be any set of $k – 1$ edges in graph *G*.
Since *G* is *k*-connected, the graph $G – S$ is 1-connected, by corollary 5.1.4.
Thus, the edge subset *S* is not an edge-cut of graph *G*,
which implies that $\kappa_e(G) \geq k$. □

Corollary 5.1.6. Let *G* be a connected graph. Then $\kappa_v(G) \leq \kappa_e(G) \leq \delta_{min}(G)$.

This is a combination of Proposition 5.1.1 and Corollary 5.1.5. □

# Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

A communications network is said to be *fault-tolerant* if it has at least two alternative paths between each pair of vertices.
This notion characterizes 2-connected graphs.
A more general result for *k*-connected graphs follows later.

Terminology: A vertex of a path *P* is an **internal vertex** of *P* if it is neither the initial nor the final vertex of that path.

Definition: Let *u* and *v* be two vertices in a graph *G*.
A collection of *u*-*v* paths in *G* is said to be **internally disjoint**
if no two paths in the collection have an internal vertex in common.

# Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

Theorem 5.1.7 [Whitney, 1932] Let $G$ be a connected graph with $n \geq 3$ vertices. Then $G$ is 2-connected if and only if for each pair of vertices in $G$, there are two internally disjoint paths between them.

Proof: ($\Leftarrow$) Suppose that graph $G$ is not 2-connected. Then let $v$ be a cut-vertex of $G$. Since $G - v$ is not connected, there must be two vertices such that there is no $x$-$y$ path in $G - v$. It follows that $v$ is an internal vertex of every $x$-$y$ path in $G$.

($\Rightarrow$) Suppose that graph $G$ is 2-connected, and let $x$ and $y$ be any two vertices in $G$. We use induction on the distance $d(x,y)$ to prove that there are at least two vertex-disjoint $x$-$y$ paths in $G$.

If there is an edge $e$ joining vertices $x$ and $y$, (i.e., $d(x,y) = 1$), then the edge-deletion subgraph $G - e$ is connected, by Corollary 5.1.4.

Thus, there is an $x$-$y$ path $P$ in $G - e$.

It follows that path $P$ and edge $e$ are two internally disjoint $x$-$y$ paths in $G$.

# Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

Next, assume for some $k \geq 2$ that the assertion holds for every pair of vertices whose distance apart is less than $k$. Let $x$ and $y$ be vertices such that distance $d(x,y) = k$, and consider an $x$-$y$ path of length $k$.

Let $w$ be the vertex that immediately precedes vertex $y$ on this path, and let $e$ be the edge between vertices $w$ and $y$.

Since $d(x,w) < k$, the induction hypothesis implies that there are two internally disjoint $x$-$w$ paths in $G$, say $P$ and $Q$.
Also, since $G$ is 2-connected, there exists an $x$-$y$ path $R$ in $G$ that avoids vertex $w$.

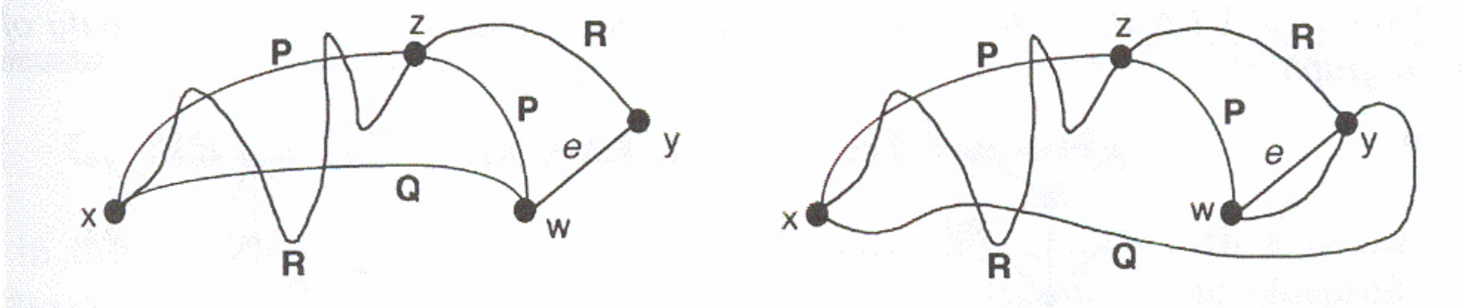



Figure 5.1.3

Path $Q$ either contains vertex $y$ (right) or it does not (left)

# Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

Let *z* be the last vertex on path *R* that precedes vertex *y* and is also on one of the paths *P* or *Q* (*z* might be vertex *x*). Assume wlog that *z* is on path *P*.

Then *G* has two internally disjoint *x*-*y* paths. One of these paths is the concatenation of the subgraph of *P* from *x* to *z* with the subpath of *R* from *z* to *y*.

If vertex *y* is not on path *Q*, then a second *x*-*y* path, internally disjoint from the first one, is the concatenation of path *Q* with the edge *e* joining vertex *w* to vertex *y*.

If *y* is on path *Q*, then the subpath of *Q* from *x* to *y* can be used as the second path.
□

# Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

Corollary 5.1.8. Let *G* be a graph with at least three vertices.
Then *G* is 2-connected if and only if any two vertices of *G* lie on a common cycle.

Proof: this follows from 5.1.7., since two vertices *x* and *y* lie on a common cycle if and only if there are two internally disjoint *x*-*y* paths.☐

# Separating set

A feasible solution to one of the problems provides a bound for the optimal value of the other problem (referred to as *weak duality*), and the optimal value of one problem is equal to the optimal value of the other (*strong duality*).

Definition: Let *u* and *v* be distinct vertices in a connected graph G.
A vertex subset (or edge subset) *S* is *u*-*v* **separating** (or **separates** *u* and *v*), if the vertices *u* and *v* lie in different components of the deletion subgraph *G* – *S.*

→ a *u*-*v* separating vertex set is a vertex-cut, and
a *u*-*v* separating edge set is an edge-cut.

When the context is clear, the term *u*-*v* **separating set** will refer either to a *u*-*v* separating vertex set or to a *u*-*v* separating edge set.

# Example

For the graph *G* in the Figure below, the vertex-cut {*x,w,z*} is a *u-v* separating set of vertices of minimum size, and the edge-cut {*a,b,c,d,e*} is a *u-v* separating set of edges of minimum size.

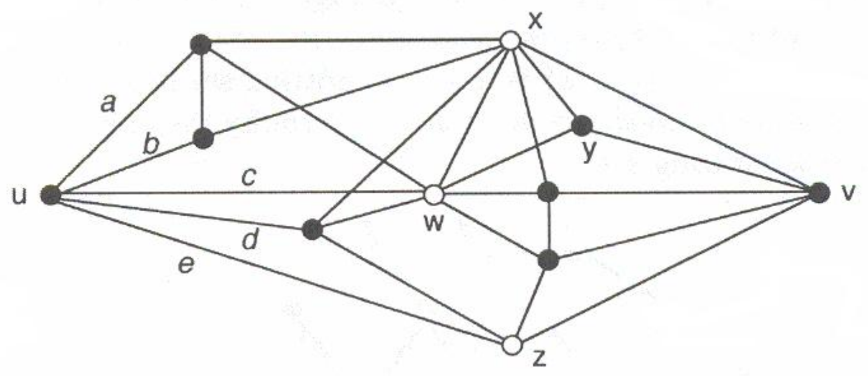



**Figure 5.3.1** Vertex- and edge-cuts that are *u-v* separating sets.

Notice that a minimum-size *u-v* separating set of edges (vertices) need not be a minimum-size edge-cut (vertex-cut).

E.g., the set {*a,b,c,d,e*} is not a minimum-size edge-cut in *G,* because the set of edges incident on the 3-valent vertex *y* is an edge-cut of size 3.

# A Primal-Dual Pair of Optimization Problems

The connectivity of a graph may be interpreted in two ways.
One interpretation is the number of vertices or edges it takes to disconnect the graph, and the other is the number of alternative paths joining any two given vertices of the graph.

Corresponding to these two perspectives are the following two optimization problems for two non-adjacent vertices *u* and *v* of a connected graph *G*.

<u>Maximization Problem</u>: Determine the maximum number of internally disjoint *u*-*v* paths in graph *G*.

<u>Minimization Problem</u>: Determine the minimum number of vertices of graph *G* needed to separate the vertices *u* and *v.*

# A Primal-Dual Pair of Optimization Problems

Proposition 5.3.1: (*Weak Duality*) Let *u* and *v* be any two non-adjacent vertices of a connected graph *G*. Let $\mathcal{P}_{uv}$ be a collection of internally disjoint *u*-*v* paths in *G*, and let $S_{uv}$ be a *u*-*v* separating set of vertices in G.

Then $|\mathcal{P}_{uv}| \leq |S_{uv}|$.

Proof: Since $S_{uv}$ is a *u*-*v* separating set, each *u*-*v* path in $\mathcal{P}_{uv}$ must include at least one vertex of $S_{uv}$. Since the paths in $\mathcal{P}_{uv}$ are internally disjoint, no two paths of them can include the same vertex.

Thus, the number of internally disjoint *u*-*v* paths in *G* is at most $|S_{uv}|$. □

Corollary 5.3.2. Let *u* and *v* be any two non-adjacent vertices of a connected graph *G*. Then the maximum number of internally disjoint *u*-*v* paths in *G* is less than or equal to the minimum size of a *u*-*v* separating set of vertices in *G*.

Menger‘s theorem will show that the two quantities are in fact equal.

# A Primal-Dual Pair of Optimization Problems

The following corollary follows directly from Proposition 5.3.1.

Corollary 5.3.3: (*Certificate of Optimality*) Let *u* and *v* be any two non-adjacent vertices of a connected graph *G.*

Suppose that $\mathcal{P}_{uv}$ is a collection of internally disjoint *u-v* paths in *G,*

and that $S_{uv}$ is a *u-v* separating set of vertices in G, such that $| \mathcal{P}_{uv} | = | S_{uv} |$.

Then $\mathcal{P}_{uv}$ is a maximum-size collection of internally disjoint *u-v* paths, and $S_{uv}$ is a minimum-size *u-v* separating set (i.e. *S* has the smallest size of all *u-v* separating sets).

# Vertex- and Edge-Connectivity

Example: In the graph *G* below, the vertex sequences ⟨*u,x,y,t,v*⟩, ⟨*u,z,v*⟩, and ⟨*u,r,s,v*⟩ represent a collection $\mathcal{P}$ of three internally disjoint *u-v* paths in *G*, and the set *S* = {*y,s,z*} is a *u-v* separating set of size 3.

Therefore, by Corollary 5.3.3, $\mathcal{P}$ is a maximum-size collection of internally disjoint *u-v* paths, and *S* is a minimum-size *u-v* separating set.

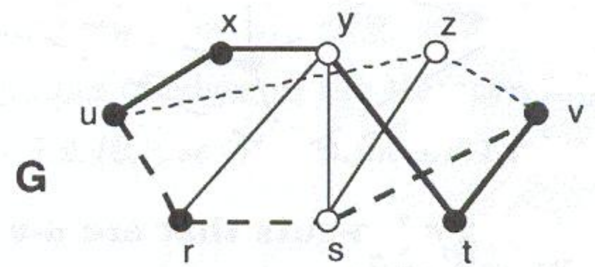


Figure 5.3.2

The theorem proved by K. Menger in 1927 (see V12) establishes a *strong duality* between the two optimization problems introduced earlier.

# strict paths

<u>Definition</u> Let *W* be a set of vertices in a graph *G* and *x* another vertex not in *W.*
A **strict *x-W* path** is a path joining *x* to a vertex in *W* and
containing no other vertex of *W.*
A **strict *W-x* path** is the reverse of a strict *x-W* path (i.e. its sequence of vertices and edges is in reverse order).

<u>Example</u>: Let us consider the ***u-v*** **separating set** *W* = *{y,s,z}* in the graph below.

There are four **strict *u-W* paths** ⟨*u,x,y*⟩, ⟨*u,r,y*⟩, ⟨*u,r,s*⟩, ⟨*u,z*⟩
And three **strict *W-v* paths** ⟨*z,v*⟩, ⟨*y,t,v*⟩, and ⟨*s,v*⟩.

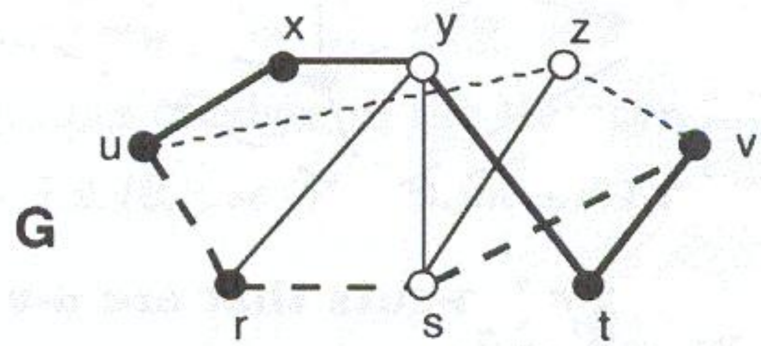


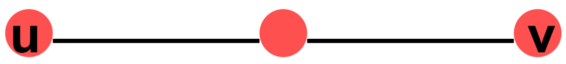

Figure 5.3.2

# Menger's Theorem

Theorem 5.3.4 [Menger, 1927] Let *u* and *v* be distinct, non-adjacent vertices in a connected graph *G.*
Then the maximum number of internally disjoint *u-v* paths in *G* equals the minimum number of vertices needed to separate *u* and *v.*

Proof: The proof uses induction on the number of edges.

The smallest graph that satisfies the premises of the theorem (non-adjacent *u* and *v*) is the path graph from *u* to *v* of length 2.




The theorem is trivially true for this graph : one cut-vertex, one *u-v* path.

# Menger's Theorem

Assume now that the theorem is true for all connected graphs having fewer than $m$ edges, e.g. for some $m \geq 3$.

Suppose that $G$ is a connected graph with $m$ edges, and let $k$ be the minimum number of vertices needed to separate the vertices $u$ and $v$.

By Corollary 5.3.2 (number of paths ≤ number of vertices),
it suffices to show that there exist $k$ internally disjoint $u$-$v$ paths in $G$.

This is clearly true if $k = 1$ (since $G$ is connected, there exists a $u$-$v$ path).

Thus, we will assume $k \geq 2$.

# Proof of Menger's Theorem

<u>Assertion 5.3.4a</u> If *G* contains a *u-v* path of length 2,
then *G* contains *k* internally disjoint *u-v* paths.

<u>Proof</u>: Suppose that $\mathcal{P} = \langle u,e_1,x,e_2,v \rangle$ is a path in *G* of length 2.
$G - x$ has fewer edges than *G* → by the induction hypothesis,
there are at least $k - 1$ internally disjoint $u - v$ paths in $G - x$.
Path $\mathcal{P}$ is internally disjoint from any of these, and, hence,
there are *k* internally disjoint *u-v* paths in *G.* □

If there is a *u-v* separating set that contains a vertex adjacent to *both* vertices *u* and *v*, then Assertion 5.3.4a guarantees the existence of *k* internally disjoint *u-v* paths in *G.*

The argument for *distance (u,v)* ≥ 3 is now broken into two cases, according to the kinds of *u-v* separating sets that exist in *G.*

# Proof of Menger's Theorem

In **Case 1** (left picture), there exists a *u*-*v* separating set *W*, where neither *u* nor *v* is adjacent to every vertex of *W* .

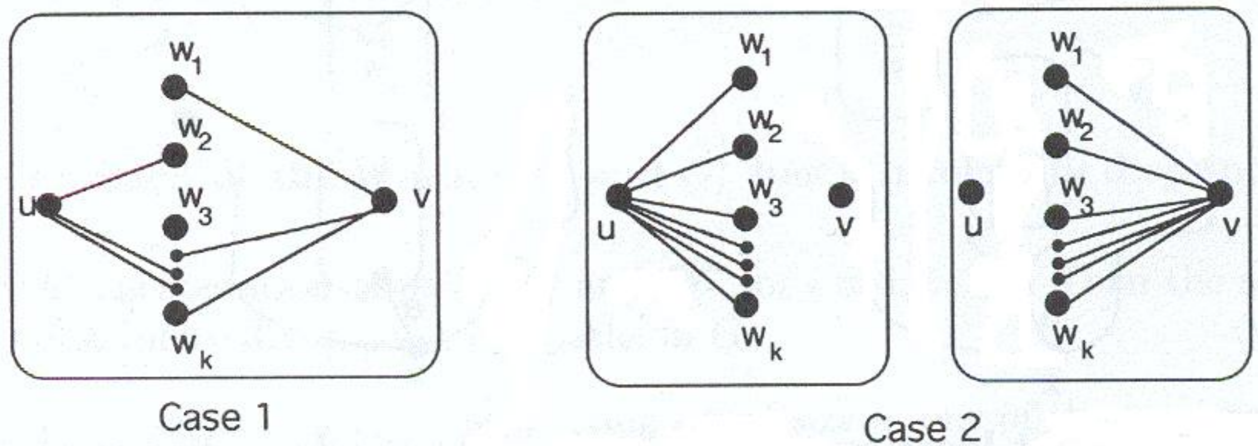



**Figure 5.3.3** The two cases remaining in the proof of Menger's theorem.

In **Case 2** (right picture), no such separating set exists.
Thus, in every *u*-*v* separating set for Case 2,
either every vertex is adjacent to *u* or every vertex is adjacent to *v*.

# Proof of Menger's Theorem

**Case 1**: There exists a *u-v* separating set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in *G* of minimum size *k,* such that neither *u* nor *v* is adjacent to every vertex in *W.*

Let $G_u$ be the subgraph induced on the union of the edge-sets of all strict *u-W* paths in *G,*
and let $G_v$ be the subgraph induced on the union of edge-sets of all strict *W-v* paths (see Fig. below).

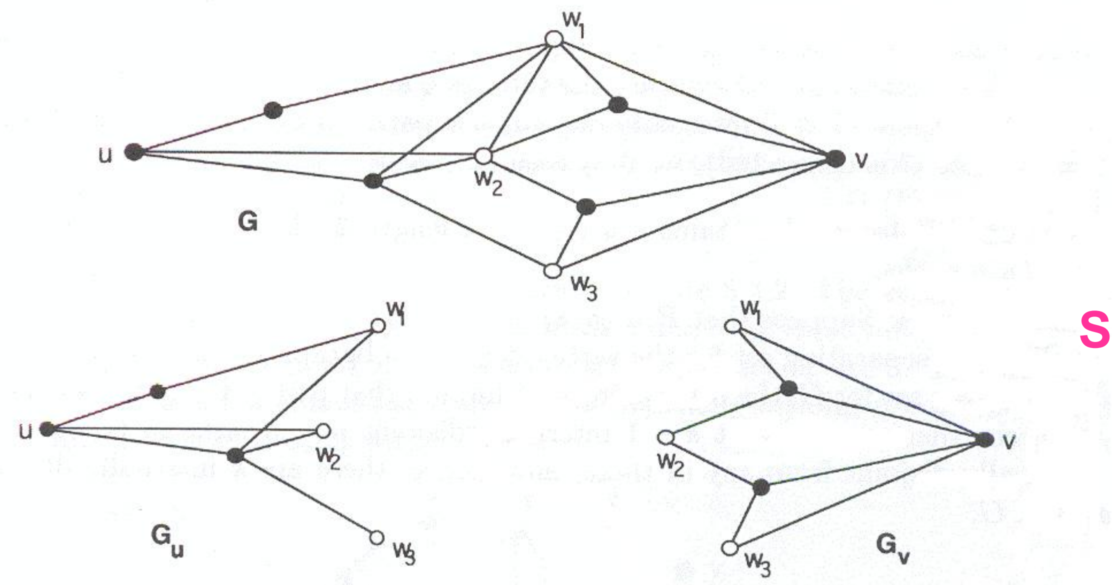



Figure 5.3.4 An example illustrating the subgraphs $G_u$ and $G_v$.

**Split up graph**

# Proof of Menger's Theorem

Assertion 5.3.4b: Both of the subgraphs $G_u$ and $G_v$ have more than $k$ edges.

Proof : For each $w_i \in W$, there is a $u$-$v$ path $P_{wi}$ in $G$
on which $w_i$ is the only vertex of $W$.
(Otherwise, $W - \{w_i\}$ would still be a $u$-$v$ separating set, which would contradict the minimality of $W$).

The $u$-$w_i$ subpath of $P_{wi}$ is a strict $u$-$W$ path that ends at $w_i$.
Thus, the final edge of this strict $u$-$W$ path is different for each $w_i$.
Hence, $G_u$ has at least $k$ edges.

The only way $G_u$ could have exactly $k$ edges would be if each of these
Strict $u$-$W$ paths consisted of a single edge joining $u$ and $w_i$, $i = 1, ..., k$.
But this is ruled out by the condition for Case 1.
Therefore, $G_u$ has more than $k$ edges.

A similar argument shows that $G_v$ also has more than $k$ edges. □

# Proof of Menger's Theorem

Assertion 5.3.4c: The subgraphs $G_u$ and $G_v$ have no edges in common.

Proof of 5.3.4c: By way of contradiction, suppose that the subgraphs $G_u$ and $G_v$ have an edge *e* in common.

By the definitions of $G_u$ and $G_v$, edge *e* would then be an edge of both a strict *u*-*W* path and a strict *W*-*v* path.

Hence, at least one of the endpoints of *e*, say *x*, is not a vertex in the *u*-*v* separating set *W* (see Fig. below).
This implies the existence of a *u*-*v* path in *G*-*W*,
which contradicts the definition of *W*. □

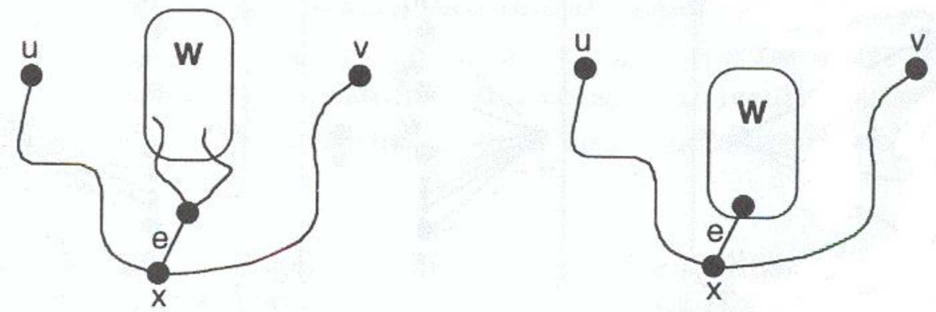



**Figure 5.3.5** At least one of the endpoints of edge *e* lies outside *W*.

# Proof of Menger's Theorem

We now define two auxiliary graphs $G_u^*$ and $G_v^*$:

***$G_u^*$*** is obtained from $G$ by replacing the subgraph $G_v$ with **a new vertex *$v^*$***
and drawing an edge from each vertex in $W$ to $v^*$, and

***$G_v^*$*** is obtained by replacing $G_u$ with **a new vertex *$u^*$***
and drawing an edge from $u^*$ to each vertex in $W$ (see Fig. below).

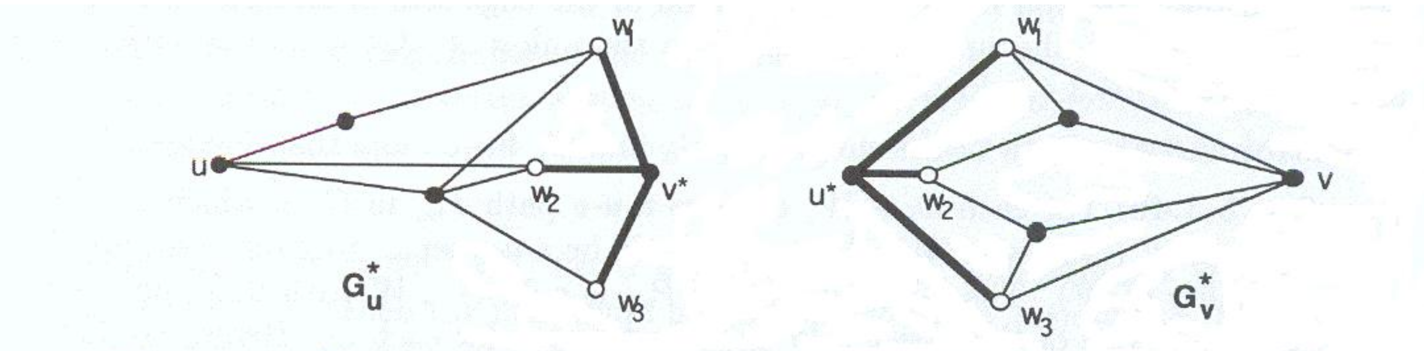



Figure 5.3.6 Illustration for the construction of graphs $G_u^*$ and $G_v^*$.

# Proof of Menger's Theorem

Assertion 5.3.4d: Both of the auxiliary graphs $G_u^*$ and $G_v^*$ have fewer edges than *G.*

*Q: Why would this be useful?*

Proof of 5.3.4d: The following chain of inequalities shows that graph $G_u^*$ has fewer edges than *G.*

$$|E_G| \geq |E_{G_u \cup G_v}|$$ since $G_u \cup G_v$ is a subgraph of $G$

$$= |E_{G_u}| + |E_{G_v}|$$

5.3.4c

$$> |E_{G_u}| + k$$

5.3.4b

$$= |E_{G_u^*}|$$ by the construction of $G_u^*$

A similar argument shows that $G_v^*$ also has fewer edges than *G.* ☐

By the construction of graphs $G_u^*$ and $G_v^*$, every $u$-$v^*$ separating set in graph $G_u^*$ and every $u^*$-$v$ separating set in graph $G_v^*$ is a $u$-$v$ separating set in graph *G.*

Hence, the set *W* is a **smallest $u$-$v^*$ separating set** in $G_u^*$ and a smallest $u^*$-$v$ separating set in $G_v^*$.

# Proof of Menger's Theorem

Since $G_u^*$ and $G_v^*$ have **fewer edges** than $G$, the **induction hypothesis** implies the existence of two collections, $\mathcal{P}_u^*$ and $\mathcal{P}_v^*$ of $k$ internally disjoint $u$-$v^*$ paths in $G_u^*$ and $k$ internally disjoint $u^*$-$v$ paths in $G_v^*$, respectively (see Fig.).

For each $w_i$, one of the paths in $\mathcal{P}_u^*$ consists of a $u$-$w_i$ path $P_i'$ in $G$ plus the new edge from $w_i$ to $v^*$, and one of the paths in $\mathcal{P}_v^*$ consists of the new edge from $u^*$ to $w_i$ followed by a $w_i$-$v$ path $P_i''$ in $G$.

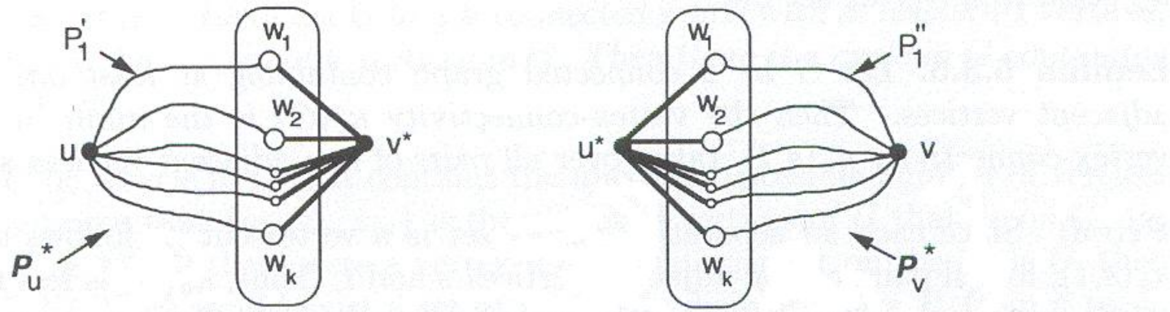



Figure 5.3.7 Each of the graphs $G_u^*$ and $G_v^*$ has $k$ internally disjoint paths.

Let $P_i$ be the **concatenation** of paths $P_i'$ and $P_i''$, for $i = 1, ..., k$.
Then the set $\{P_i\}$ is a collection of $k$ internally disjoint $u$-$v$ paths in $G$. ☐ (Case 1)

# Proof of Menger's Theorem

**Case 2**: Suppose that for each *u*-*v* separating set of size *k*, one of the vertices *u* or *v* is adjacent to all the vertices in that separating set.

**will not be proven in lecture**

Let $P = \langle u, e_1, x_1, e_2, x_2, ..., v \rangle$ be a shortest *u*-*v* path in *G*.
By Assertion 5.3.4a, we can assume that *P* has length at least 3 and that vertex $x_1$ is not adjacent to vertex *v*.
By Proposition 5.1.3, the edge-deletion subgraph $G - e_2$ is connected.
Let *S* be a smallest *u*-*v* separating set in subgraph $G - e_2$ (see Fig.).

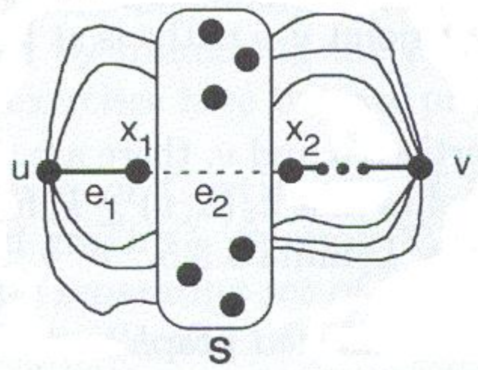



Figure 5.3.8 Completing Case 2 of Menger's theorem.

# Proof of Menger's Theorem

Then $S$ is a $u$-$v$ separating set in the vertex-deletion subgraph $G - x_1$.
Thus, $S \cup \{x_1\}$ is a $u$-$v$ separating set in $G$, which implies that $|S| \geq k - 1$, by the minimality of $k$. On the other hand, the minimality of
$|S|$ in $G - e_2$ implies that $|S| \leq k$, since every $u$-$v$ separating set in $G$ is also a $u$-$v$ separating set in $G - e_2$.

If $|S| = k$, then, by the induction hypothesis, there are $k$ internally disjoint $u$-$v$ paths in $G - e_2$ and, hence, in $G$.
If $|S| = k - 1$, then $x_i \notin S$, $i = 1,2$ (otherwise $S - \{x_i\}$ would be a $u$-$v$ separating set in $G - e_2$, contradicting the minimality of $k$).
Thus, the sets $S \cup \{x_1\}$ and $S \cup \{x_2\}$ are both of size $k$ and both $u$-$v$ separating sets of $G$. The condition for Case 2 and the fact that vertex $x_1$ is not adjacent to $v$ imply that every vertex in $S$ is adjacent to vertex $u$.
Hence, no vertex in $S$ is adjacent to $v$ (lest there be a $u$-$v$ path of length 2).
But then the condition of Case applied to $S \cup \{x_2\}$ implies that vertex $x_2$ is adjacent to vertex $u$, which contradicts the minimality of path $P$ and completes the proof. □