V12 Solving the Maximum-Flow Problem

We will present an algorithm that originated by Ford and Fulkerson (1962).
Idea: increase the flow in a network iteratively until it cannot be increased any further → **augmenting flow path**.

Suppose that *f* is a flow in a capacitated *s-t* network *N,* and suppose that there exists a directed *s-t* path

$$P = \langle s, e_1, v_1, e_2, \ldots, e_k, t \rangle$$

in *N*, such that $f(e_i) < cap(e_i)$ for $i=1, \ldots, k$.

Then considering arc capacities only, the flow on each arc e_i can be increased by as much as $cap(e_i) - f(e_i)$.

But to maintain the conservation-of-flow property at each of the vertices v_i, the increases on all of the arcs of path *P* must be equal.

Thus, if Δ_P denotes this increase, then the largest possible value for Δ_P is $min\{cap(e_i\} - f(e_i)\}$.

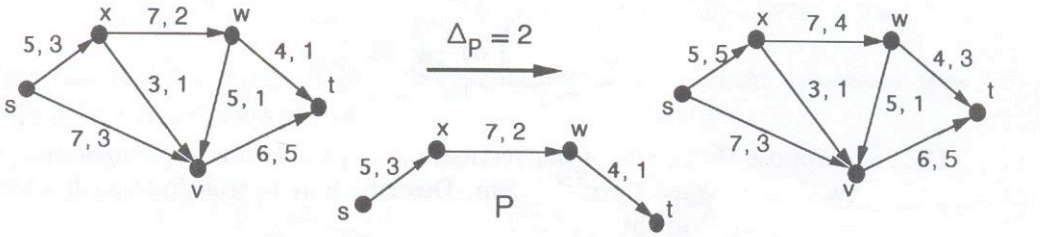
Solving the Maximum-Flow Problem

Example: Left side: the value of the current flow is 6.

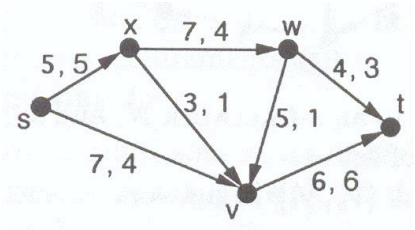
Consider the directed *s-t* path $P = \langle s,x,w,t \rangle$.

The flow on each arc of path P can be increased by $\Delta_P = 2$.

The resulting flow, which has value 8, is shown on the right side.



Using the directed path $\langle s,v,t \rangle$, the flow can be increased to 9. The resulting flow is shown right.

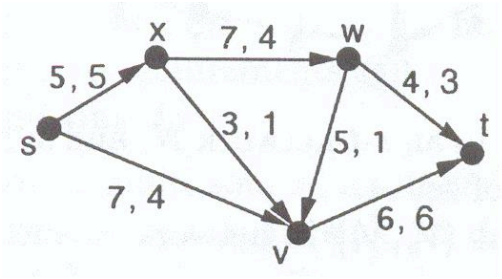


At this point, the flow cannot be increased any further along **directed *s-t paths***, because each such path must either use the arc directed from *s* to *x* or from *v* to *t*. Both arcs have flow at capacity.

Solving the Maximum-Flow Problem

However, the flow can be increased further.

E.g. increase the flow on the arc from source *s* to vertex *v* by one unit, decrease the flow on the arc from *w* to *v* by one unit, and increase the flow on the arc from *w* to *t* by one unit.



f-Augmenting Paths

Definition: An *s-t* **quasi-path** in a network *N* is an alternating sequence

$$\langle s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t \rangle$$

of vertices and arcs that forms an *s-t* path in the underlying undirected graph of *N.*

Terminology For a given *s-t* quasi-path

$$Q = \langle s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t \rangle$$

arc e_i is called a **forward arc** if it is directed from vertex v_{i-1} to vertex v_i and arc e_i is called a **backward arc** if it is directed from v_i to v_{i-1}.

Clearly, a directed *s-t* path is a quasi-path whose arcs are all forward.

Example. On the *s-t* quasi-path shown below, arcs *a* and *b* are backward, and the three other arcs are forward.

f-Augmenting Paths

Definition: Let *f* be a flow in an *s-t* network *N.* An ***f*-augmenting path** Q is an *s-t* quasi path in *N* such that the flow on each forward arc can be increased, and the flow on each backward arc can be decreased.

Thus, for each arc *e* on an *f*-augmenting path Q,

$f(e) < cap(e),$	if *e* is a forward arc
$f(e) > 0$	if *e* is a backward arc.

Notation For each arc *e* on a given *f*-augmenting path Q, let Δ_e be the quantity given by

$$\Delta_e = \begin{cases} cap(e) - f(e), & \text{if } e \text{ is a forward arc} \\ f(e), & \text{if } e \text{ is a backward arc} \end{cases}$$

Terminology The quantity Δ_e is called the **slack on arc** *e.* Its value on a forward arc is the largest possible increase in the flow, and on a backward arc, the largest possible decrease in the flow, disregarding conservation of flow.

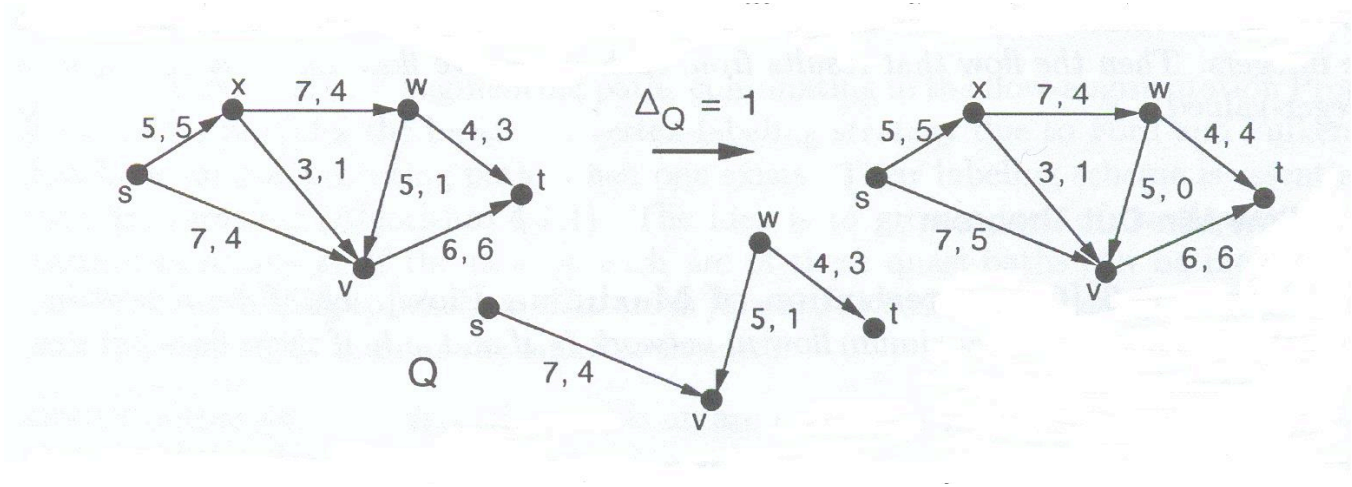
f-Augmenting Paths

Remark Conservation of flow requires that the change in the flow on the arcs of an augmenting flow path be of equal magnitude.

Thus, the maximum allowable change in the flow on an arc of quasipath Q is Δ_Q, where

$$\Delta_Q = \min_{e \in Q} \{\Delta_e\}$$

Example For the example network shown below, the current flow *f* has value 9, and the quasi-path Q = ⟨*s,v,w,t*⟩ is an *f*-augmenting path with $\Delta_Q = 1$.



flow augmentation

Proposition 12.2.1 (Flow Augmentation) Let *f* be a flow in a network *N*, and let Q be an *f*-augmenting path with minimum slack Δ_Q on its arcs.
Then the augmented flow *f'* given by

$$f'(e) = \begin{cases} f(e) + \Delta_Q, & \text{if } e \text{ is a forward arc of Q} \\ f(e) - \Delta_Q, & \text{if } e \text{ is a backward arc of Q} \\ f(e) & \text{otherwise} \end{cases}$$

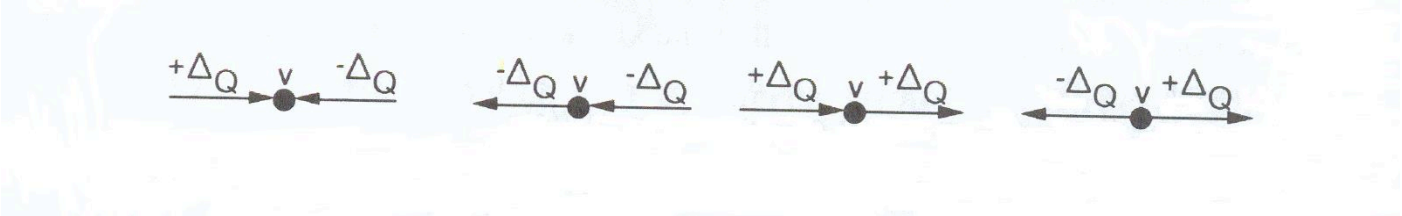
is a feasible flow in network *N* and *val(f')* = *val(f)* + Δ_Q.

Proof. Clearly, $0 \leq f'(e) \leq cap(e)$, by the definition of Δ_Q.

The only vertices through which the net flow may have changed are those vertices on the augmenting path Q. Thus, to verify that *f'* satisfies conservation of flow, only the internal vertices of Q need to be checked.

f-Augmenting Paths

For a given vertex *v* on augmenting path Q, the two arcs of Q that are incident on *v* are configured in one of four ways, as shown below. In each case, the net flow into or out of vertex *v* does not change, thereby preserving the conservation-of-flow property.



It remains to be shown that the flow has increased by Δ_Q.

The only arc incident on the source *s* whose flow has changed is the first arc e_1 of augmenting path Q.

If e_1 is a forward arc, then $f'(e_1) = f(e_1) + \Delta_Q$, and

if e_1 is a backward arc, then $f'(e_1) = f(e_1) - \Delta_Q$. In either case,

$$val(f') = \sum_{e \in Out(s)} f'(e) - \sum_{e \in In(s)} f'(e) = \Delta_Q + val(f) \quad \square$$

Max-Flow Min-Cut

Theorem 12.2.3 [Characterization of Maximum Flow]

Let *f* be a flow in a network *N.*

Then *f* is a maximum flow in network *N* if and only if there does not exist an *f*-augmenting path in *N.*

Proof: Necessity ($\Rightarrow$) Suppose that *f* is a maximum flow in network *N*.

Then by Proposition 12.2.1, there is no *f*-augmenting path.

Proposition 12.2.1 (Flow Augmentation) Let *f* be a flow in a network *N*, and let Q be an *f*-augmenting path with minimum slack ΔQ on its arcs. Then the augmented flow *f'* given by

$$f'(e) = \begin{cases} f(e) + \Delta_Q, & \text{if } e \text{ is a forward arc of Q} \\ f(e) - \Delta_Q, & \text{if } e \text{ is a backward arc of Q} \\ f(e) & \text{otherwise} \end{cases}$$

is a feasible flow in network *N* and *val(f')* = *val(f)* + ΔQ.

$\rightarrow$ assuming an *f*-augmenting path existed, we could construct a flow *f'* with *val(f')* > *val(f)* contradicting the maximality of *f.*

Max-Flow Min-Cut

Sufficiency ($\Leftarrow$) Suppose that there does not exist an *f*-augmenting path in network *N.*

Consider the collection of all quasi-paths in network *N* that begin with source *s*, and have the following property: each forward arc on the quasi-path has positive slack, and each backward arc on the quasi-path has positive flow.

Let V_s be the union of the vertex-sets of these quasi-paths.

Since there is no *f*-augmenting path, it follows that sink $t \notin V_s$.

Let $V_t = V_N - V_s$.

Then $\langle V_s, V_t \rangle$ is an *s*-*t* cut of network *N*. Moreover, by definition of the sets V_s and V_t ,

$$f(e) = \begin{cases} cap(e) & \text{if } e \in \langle V_s, V_t \rangle \\ 0 & \text{if } e \in \langle V_t, V_s \rangle \end{cases}$$

(if the flow along these edges *e* were not cap(e) or 0, these edges would belong to V_s!)

Hence, *f* is a maximum flow, by Corollary 12.1.8. □

Max-Flow Min-Cut

Theorem 12.2.4 [Max-Flow Min-Cut] For a given network, the value of a maximum flow is equal to the capacity of a minimum cut.

Proof: The *s-t* cut $\langle V_s, V_t \rangle$ that we just constructed in the proof of Theorem 12.2.3 (direction $\Leftarrow$) has capacity equal to the maximum flow. □

The outline of an algorithm for maximizing the flow in a network emerges from Proposition 12.2.1 and Theorem 12.2.3.

Algorithm 12.2.1: Outline for Maximum Flow

Input: an *s-t* network N.
Output: a maximum flow f^* in network N.

[*Initialization*]
For each arc e in network N
  $f^*(e) := 0$
[*Flow Augmentation*]
While there exists an f^*-augmenting path in network N
  Find an f^*-augmenting path Q.
  Let $\Delta_Q = \min_{e \in Q}\{\Delta_e\}$.
  For each arc e of augmenting path Q
    If e is a forward arc
      $f^*(e) := f^*(e) + \Delta_Q$
    Else (e is a backward arc)
      $f^*(e) := f^*(e) - \Delta_Q$
Return flow f^*.

Finding an *f*-Augmenting Path

The discussion of *f*-augmenting paths culminating in the flow-augmenting Proposition 12.2.1 provides the basis of a vertex-labeling strategy due to Ford and Fulkerson that finds an *f*-augmenting path, when one exists.

Their labelling scheme is essentially **basic tree-growing**.

The idea is to grow a tree of quasi-paths, each starting at source *s*.

If the flow on each arc of these quasi-paths can be increased or decreased, according to whether that arc is **forward** or **backward**, then an *f*-augmenting path is obtained as soon as the sink *t* is labelled.

Finding an *f*-Augmenting Path

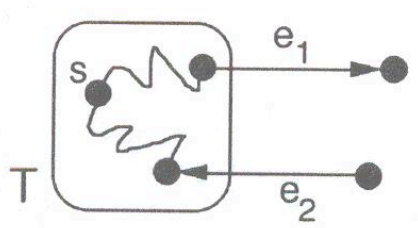
A **frontier arc** is an arc *e* directed from a **labeled** endpoint *v* to an **unlabeled** endpoint *w*.

For constructing an *f*-augmenting path, the frontier path *e* is allowed to be backward (directed from vertex *w* to vertex *v*), and it can be added to the tree as long as it has slack $\Delta_e > 0$.

Finding an *f*-Augmenting Path

Terminology: At any stage during tree-growing for constructing an *f*-augmenting path, let *e* be a frontier arc of tree *T*, with endpoints *v* and *w.*
The arc *e* is said to be **usable** if, for the current flow *f*, either

e is directed from vertex *v* to vertex *w* and $f(e) < cap(e)$, or
e is directed from vertex *w* to vertex *v* and $f(e) > 0$.



Frontier arcs e_1 and e_2 are usable if
$f(e_1) < cap(e_1)$ and $f(e_2) > 0$

Remark From this vertex-labeling scheme, any of the existing *f*-augmenting paths could result. But the efficiency of Algorithm 12.2.1 is based on being able to find „good“ augmenting paths.
If the arc capacities are irrational numbers, then an algorithm using the Ford&Fulkerson labeling scheme might not terminate (strictly speaking, it would not be an algorithm).

Finding an *f*-Augmenting Path

Even when flows and capacities are restricted to be integers, problems concerning efficiency still exist.

E.g., if each flow augmentation were to increase the flow by only one unit, then the number of augmentations required for maximization would equal the capacity of a minimum cut.

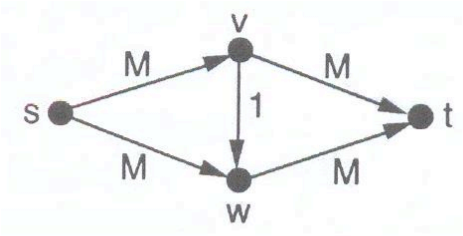
Such an algorithm would depend on the size of the arc capacities instead of on the size of the network.

Finding an *f*-Augmenting Path

Example: For the network shown below, the arc from vertex *v* to vertex *w* has flow capacity 1, while the other arcs have capacity *M*, which could be made arbitrarily large.

If the choice of the augmenting flow path at each iteration were to alternate between the directed path $\langle s,v,w,t \rangle$ and the quasi path $\langle s,w,v,t \rangle$, then the flow would increase by only one unit at each iteration.

Thus, it could take as many as $2M$ iterations to obtain the maximum flow.



Finding an *f*-Augmenting Path

Edmonds and Karp avoid these problems with this algorithm.

It uses **breadth-first search** to find an *f*-augmenting path with the smallest number of arcs.

Algorithm 12.2.2: Finding an Augmenting Path

Input: a flow f in an s-t network N.
Output: an f-augmenting path Q or a minimum s-t cut with capacity $val(f)$.

Initialize vertex set $V_s := \{s\}$.
Write label 0 on vertex s.
Initialize label counter $i := 1$
While vertex set V_s does not contain sink t
    If there are usable arcs
        Let e be a usable arc whose labeled endpoint v
        has the smallest possible label.
        Let w be the unlabeled endpoint of arc e.
        Set $backpoint(w) := v$.
        Write label i on vertex w.
        $V_s := V_s \cup \{w\}$
        $i := i + 1$
    Else
        Return s-t cut $\langle V_s, V_N - V_s \rangle$.
Reconstruct the f-augmenting path Q by following backpointers,
starting from sink t.
Return f-augmenting path Q.

FFEK algorithm: Ford, Fulkerson, Edmonds, and Karp

Algorithm 12.2.3 combines Algorithms 12.2.1 and 12.2.2

Algorithm 12.2.3: FFEK - Maximum Flow

Input: an s-t network N.
Output: a maximum flow f^* in network N.

[*Initialization*]
For each arc e in network N
  $f^*(e) := 0$
[*Flow Augmentation*]
Repeat
  Apply Algorithm 12.2.2 to find an f^*-augmenting path Q.
  Let $\Delta_Q = \min_{e \in Q}\{\Delta_e\}$.
  For each arc e of augmenting path Q
    If e is a forward arc
      $f^*(e) := f^*(e) + \Delta_Q$
    Else (e is a backward arc)
      $f^*(e) := f^*(e) - \Delta_Q$
Until an f^*-augmenting path cannot be found in network N.
Return flow f^*.

FFEK algorithm: Ford, Fulkerson, Edmonds, and Karp

Example: the figures illustrate algorithm 12.2.3.

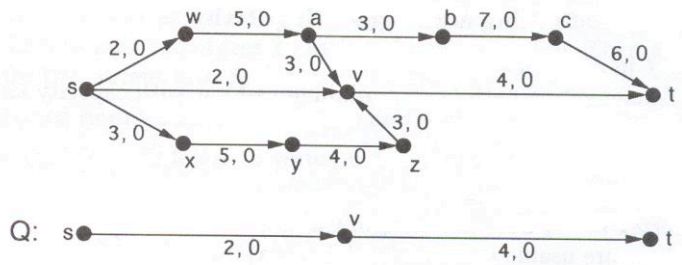


Figure 12.2.8 Iteration 0: $val(f) = 0$; augmenting path Q has $\Delta_Q = 2$.

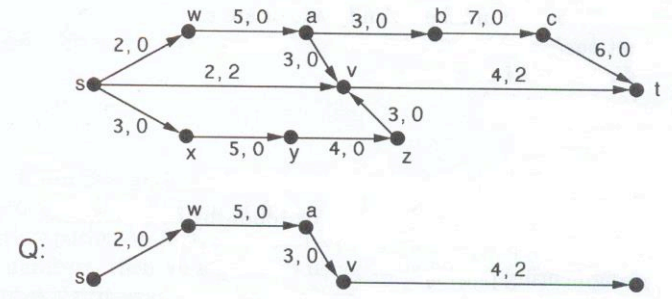


Figure 12.2.9 Iteration 1: $val(f) = 2$; augmenting path Q has $\Delta_Q = 2$.

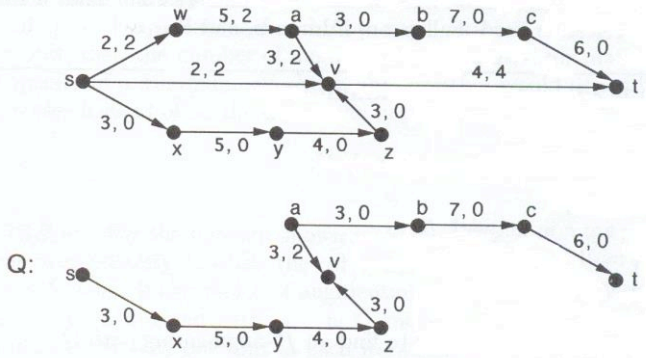
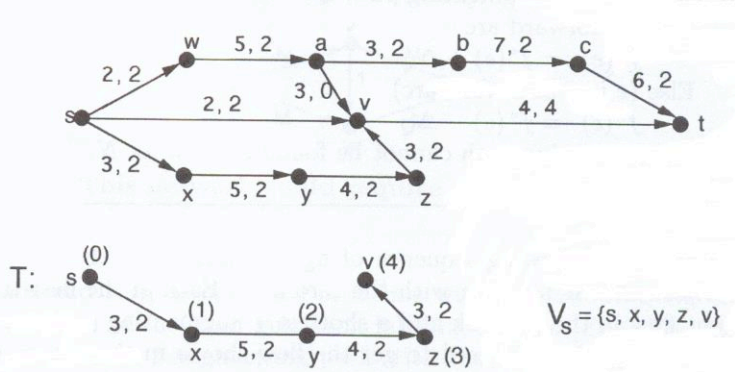


Figure 12.2.10 Iteration 2: $val(f) = 4$; augmenting path Q has $\Delta_Q = 2$.



e 12.2.11 Final iteration: $val(f) = 6 = cap\langle\{s, x, y, z, v\}, \{w, a, b, c, t\}\rangle$.

<{s, x, y, z, v}, {w, a, b, c, t}> is the *s-t* cut with capacity equal to the current flow, establishing optimality.

FFEK algorithm: Ford, Fulkerson, Edmonds, and Karp

At the end of the final iteration, the two arcs from source *s* to vertex *w* and the arc directed from vertex *v* to sink *t* form the minimum cut $\langle \{s,x,y,z,v\}, \{w,a,b,c,t\} \rangle$. Neither of them is usable, i.e. the flow(e) = cap(e).

This illustrates the *s*-*t* cut that was constructed in the proof of theorem 12.2.3.