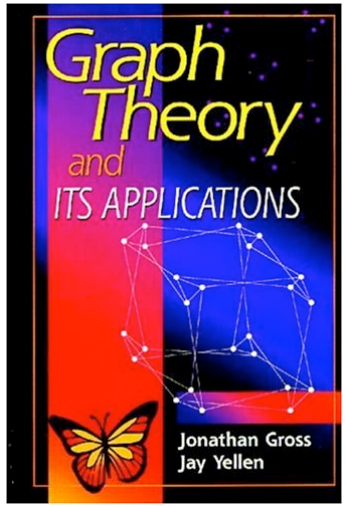
V13 Network Flows

This part follows closely chapter 12.1 in the book on the right on „Flows and Cuts in Networks and
Chapter 12.2 on “Solving the Maximum-Flow Problem“



Flow in Networks can mean
- flow of oil or water in pipelines, electricity
- phone calls, emails, traffic networks ...

Equivalences between
max-flow min-cut theorem of Ford and Fulkerson
& the connectivity theorems of Menger
→ led to the development of efficient algorithms for a number of practical problems to solve scheduling and assignment problems.

Single Source – Single Sink Capacitated Networks

Definition: A **single source – single sink network** is a connected digraph that has a distinguished vertex called the **source** with nonzero outdegree and a distinguished vertex called the **sink** with nonzero indegree.

Such a network with source *s* and sink *t* is often referred to as a ***s-t* network**.

Definition: A **capacitated network** is a connected digraph such that each arc *e* is assigned a nonnegative weight ***cap(e),*** called the **capacity** of arc *e.*

Notation: Let *v* be a vertex in a digraph *N*. Then ***Out(v)*** denotes the set of all arcs that are directed **from** vertex *v*. That is,

$$Out(v) = \{e \in E_N \mid tail(e) = v\}$$

Correspondingly, ***In(v)*** denotes the set of arcs that are directed **to** vertex *v:*

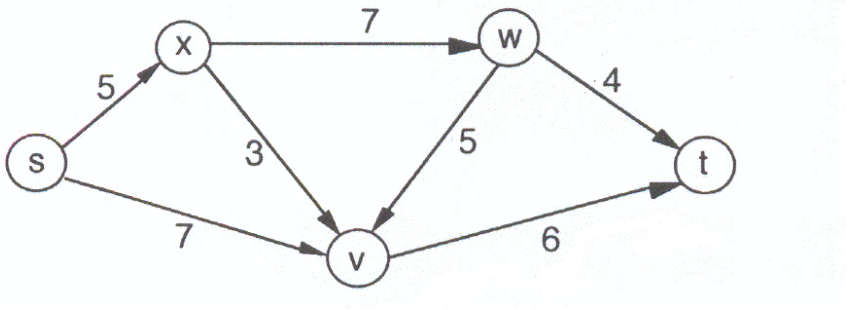
$$In(v) = \{e \in E_N \mid head(e) = v\}$$

Single Source – Single Sink Capacitated Networks

Notation: For any two vertex subsets *X* and *Y* of a digraph *N*, let $\langle X,Y \rangle$ denote the set of arcs in *N* that are directed **from** a vertex in *X* **to** a vertex in *Y*.

$$\langle X,Y \rangle = \{ e \in E_N \mid tail(e) \in X \text{ and } head(e) \in Y \}$$

Example: The figure shows a 5-vertex capacitated *s*-*t*-network.
If $X = \{x,v\}$ and $Y = \{w,t\}$, then the elements of arc set $\langle X,Y \rangle$ are the arc directed from vertex *x* to vertex *w* and the arc directed from vertex *v* to sink *t*.



A 5-vertex capacitated network with source *s* and sink *t*.

The only element in arc set $\langle Y,X \rangle$ is the arc directed from vertex *w* to vertex *v*.

Feasible Flows

Definition: Let *N* be a capacitated *s*-*t*-network.

A **feasible flow** *f* in *N* is a function $f:E_N \rightarrow R^+$ that assigns a nonnegative real number

1. (**capacity constraints**) $f(e) \leq cap(e)$, for every arc *e* in network *N.*
2. (**conservation constraints**) $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$

for every vertex *v* in network *N*, other than source *s* and sink *t.*

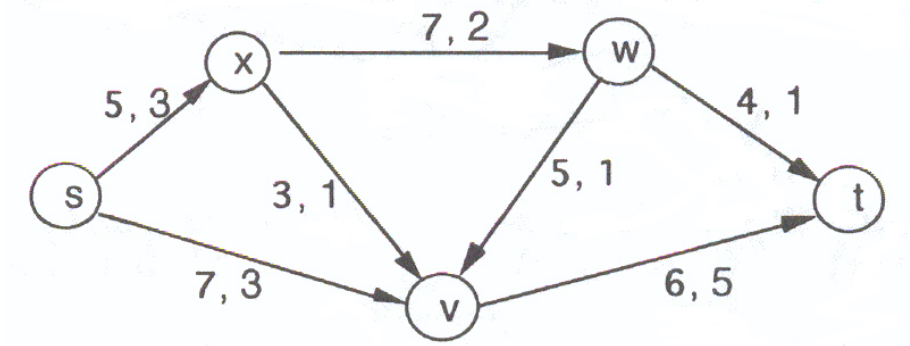
Property 2 above is called the **conservation-of-flow** condition.

E.g. for an oil pipeline, the total flow of oil going into any juncture (vertex) in the pipeline must equal the total flow leaving that juncture.

Notation: to distinguish visually between the flow and the capacity of an arc, we adopt the convention in drawings that when both numbers appear, the capacity will always be in bold and to the left of the flow.

Feasible Flows

Example: The figure shows a feasible flow for the previous network.
Notice that the total amount of flow leaving source *s* equals 6, which is also the net flow entering sink *t.*



Definition: The **value of flow** *f* in a capacitated network *N*, denoted with ***val(f)***, is the net flow leaving the source *s*, that is

$$val(f) = \sum_{e \in Out(s)} f(e) - \sum_{e \in In(s)} f(e)$$

Definition: The **maximum flow *f**** in a capacitated network *N* is a flow in *N* having the maximum value, i.e. *val(f) ≤ val(f*),* for every flow *f* in *N.*

Cuts in *s-t* Networks

By definition, any nonzero flow must use at least one of the arcs in *Out(s).*
In other words, if all of the arcs in *Out(s)* were deleted from network *N*,
then no flow could get from source *s* to sink *t.*

This is a special case of the following definition, which combines the concepts of **partition-cut** and **s-t separating set.**

From V11
Definition: Let *G* be a graph, and let X_1 and X_2 form a partition of V_G.
The set of all edges of *G* having one endpoint in X_1 and the other endpoint in X_2 is called a **partition-cut** of G and is denoted $\langle X_1, X_2 \rangle$.

From V12
Definition: Let *u* and *v* be distinct vertices in a connected graph G.
A vertex subset (or edge subset) *S* is *u-v* **separating** (or **separates** *u* and *v*), if the vertices *u* and *v* lie in different components of the deletion subgraph *G – S.*

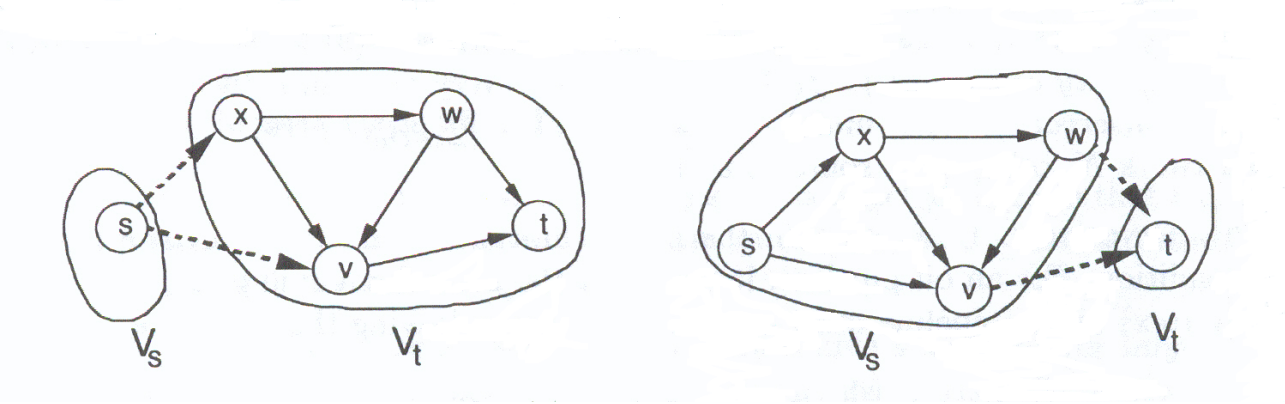
Cuts in *s-t* Networks

Definition: Let N be an *s-t* network, and let V_s and V_t form a partition of V_G such that source $s \in V_s$ and sink $t \in V_t$.
Then the set of all arcs that are directed from a vertex in set V_s to a vertex in set V_t is called an ***s-t* cut** of network N and is denoted $\langle V_s, V_t \rangle$.

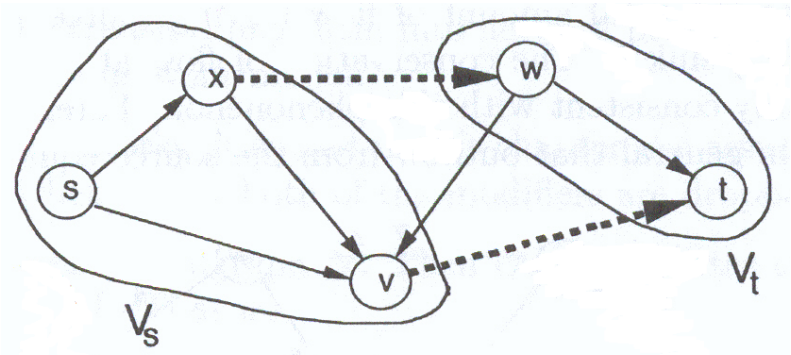
Remark: The arc set ***Out(s)*** for an *s-t* network N is the *s-t* cut $\langle \{s\}, V_N - \{s\} \rangle$, and ***In(t)*** is the *s-t* cut $\langle V_N - \{t\}, \{t\} \rangle$.

Cuts in *s-t* Networks

Example. The figure portrays the arc sets *Out(s)* and *In(t)* as *s-t* cuts, where *Out(s)* = ⟨ *{s}, {x,v,w,t}* ⟩ and *In(t)* = ⟨ {*s,x,v,w*},{*t*} ⟩.



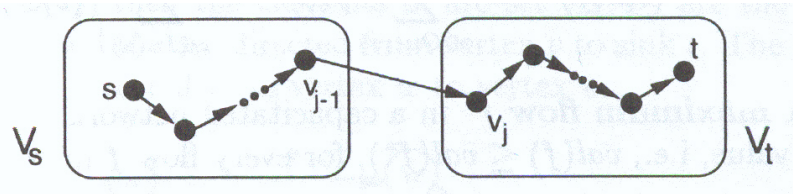
Example: a more general *s-t* cut ⟨ V_s, V_t ⟩ is shown below, where V_s = {*s,x,v*} and V_t = {*w,t*}.



Cuts in *s-t* Networks

Proposition 12.1.1 Let $\langle V_s, V_t \rangle$ be an *s-t* cut of a network *N.*
Then every directed *s-t* path in *N* contains at least one arc in $\langle V_s, V_t \rangle$.

Proof. Let $P = \langle s = v_0, v_1, v_2, \ldots, v_l = t \rangle$ be the vertex sequence of a directed *s-t* path in network *N.*
Since $s \in V_s$ and $t \in V_t$, there must be a first vertex v_j on this path that is in set V_t (see figure below).
Then the arc from vertex v_{j-1} to v_j is in $\langle V_s, V_t \rangle$. □



Relationship between Flows and Cuts

Similar to viewing the set *Out(s)* of arcs directed from source *s* as the *s*-*t* cut $\langle \{s\}, V_N - \{s\} \rangle$, the set *In(s)* may be regarded as the set of „backward“ arcs relative to this cut, namely, the arc set $\langle V_N - \{s\}, \{s\}, \rangle$.

From this perspective, the definition of *val(f)* may be rewritten as

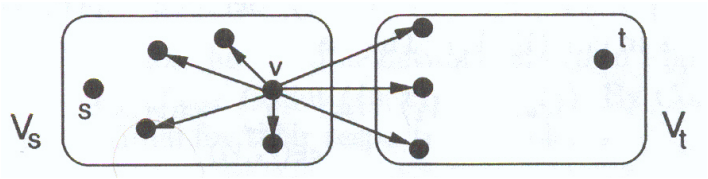
$$val(f) = \sum_{e \in \langle \{s\}, V_N - \{s\} \rangle} f(e) - \sum_{e \in \langle V_N - \{s\}, \{s\} \rangle} f(e)$$

Relationship between Flows and Cuts

Lemma 12.1.2. Let $\langle V_s, V_t \rangle$ be any *s-t* cut of an *s-t* network *N*. Then

$$\bigcup_{v \in V_s} Out(v) = \langle V_s, V_s \rangle \cup \langle V_s, V_t \rangle \quad \text{and} \quad \bigcup_{v \in V_s} In(v) = \langle V_s, V_s \rangle \cup \langle V_t, V_s \rangle$$

Proof: For any vertex $v \in V_s$, each arc directed from v is either in $\langle V_s, V_s \rangle$ or in $\langle V_s, V_t \rangle$. The figure illustrates for a vertex v the partition of *Out(v)* into a 4-element subset of $\langle V_s, V_s \rangle$ and a 3-element subset of $\langle V_s, V_t \rangle$.
Similarly, each arc directed to vertex v is either in $\langle V_s, V_s \rangle$ or in $\langle V_t, V_s \rangle$. □



$$\bigcup_{v \in V_s} Out(v) = \langle V_s, V_s \rangle \cup \langle V_s, V_t \rangle$$

Relationship between Flows and Cuts

Proposition 12.1.3. Let *f* be a flow in an *s-t* network *N,* and let $\langle V_s, V_t \rangle$ be any *s-t* cut of *N.* Then $val(f) = \sum_{e \in \langle V_s, V_t \rangle} f(e) - \sum_{e \in \langle V_t, V_s \rangle} f(e)$

Proof: By definition,

$$val(f) = \sum_{e \in Out(s)} f(e) - \sum_{e \in In(s)} f(e)$$

And by the conservation of flow

$$\sum_{e \in Out(v)} f(e) - \sum_{e \in In(v)} f(e) = 0 \quad \text{for every } v \in V_s \text{ other than } s. \text{ Thus}$$

$$val(f) = \sum_{v \in V_s} \left(\sum_{e \in Out(v)} f(e) - \sum_{e \in In(v)} f(e) \right) = \sum_{v \in V_s} \sum_{e \in Out(v)} f(e) - \sum_{v \in V_s} \sum_{e \in In(v)} f(e) \qquad (1)$$

By Lemma 12.1.2.

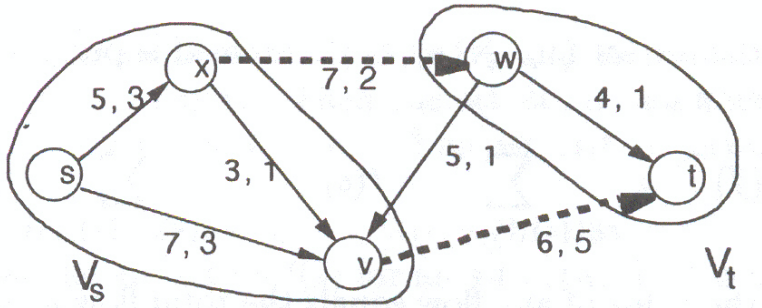
$$\sum_{v \in V_s} \sum_{e \in Out(v)} f(e) = \sum_{e \in \langle V_s, V_s \rangle} f(e) + \sum_{e \in \langle V_s, V_t \rangle} f(e) \text{ and}$$

$$\sum_{v \in V_s} \sum_{e \in In(v)} f(e) = \sum_{e \in \langle V_s, V_s \rangle} f(e) + \sum_{e \in \langle V_t, V_s \rangle} f(e) \qquad (2)$$

Now enter the right hand sides of (2) into (1) and obtain the desired equality. □

Example

The flow *f* and cut $\langle \{s,x,v\},\{w,t\} \rangle$ shown in the figure illustrate Proposition 12.1.3.



$$6 = val(f) = \sum_{e \in \langle \{s,x,v\},\{w,t\} \rangle} f(e) - \sum_{e \in \langle \{w,t\},\{s,x,v\} \rangle} f(e) = 7 - 1$$

The next corollary confirms something that was apparent from intuition: the net flow out of the source *s* equals the net flow into the sink *t*.

Corollary 12.1.4 Let *f* be a flow in an *s*-*t* network. Then

$$val(f) = \sum_{e \in In(t)} f(e) - \sum_{e \in Out(t)} f(e)$$

Proof: Apply proposition 12.1.3 to the *s*-*t* cut $In(t) = \langle V_N - \{t\}, \{t\} \rangle$. □

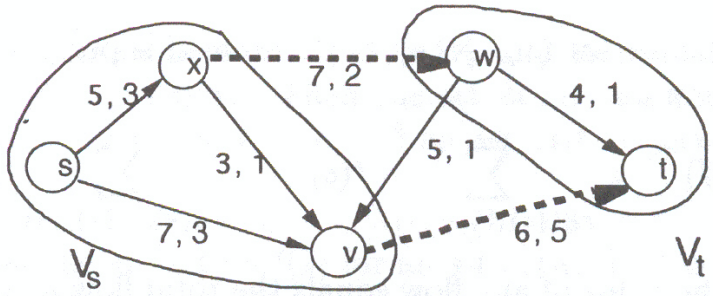
Example

Definition. The **capacity of a cut** $\langle V_s, V_t \rangle$ denoted **cap**$\langle V_s, V_t \rangle$, is the sum of the capacities of the arcs in cut $\langle V_s, V_t \rangle$. That is

$$cap\langle V_s, V_t \rangle = \sum_{e \in \langle V_s, V_t \rangle} cap(e)$$

Definition. The **minimum cut** of a network *N* is a cut with the minimum capacity.

Example. The capacity of the cut shown in the previous figure is 13,
And the cut ⟨{s,x,v,w},{t}⟩ with capacity 10, is the only minimum cut.



Maximum-Flow and Minimum-Cut Problems

The problems of finding the maximum flow in a capacitated network *N* and finding a minimum cut in *N* are closely related.

These two optimization problems form a *max-min* pair.

The following proposition provides an upper bound for the maximum-flow problem.

Maximum-Flow and Minimum-Cut Problems

Proposition 12.1.5 Let *f* be any flow in an *s*-*t* network, and let $\langle V_s, V_t \rangle$ be any *s*-*t* cut. Then $val(f) \leq cap\langle V_s, V_t \rangle$

Proof:

$$val(f) = \sum_{e \in \langle V_s, V_t \rangle} f(e) - \sum_{e \in \langle V_t, V_s \rangle} f(e) \qquad \text{(by proposition 12.1.3)}$$

$$\leq \sum_{e \in \langle V_s, V_t \rangle} cap(e) - \sum_{e \in \langle V_t, V_s \rangle} f(e) \qquad \text{(by capacity constraints)}$$

$$= cap\langle V_s, V_t \rangle - \sum_{e \in \langle V_t, V_s \rangle} f(e) \qquad \text{(by definition of } cap\langle V_s, V_t \rangle\text{)}$$

$$\leq cap\langle V_s, V_t \rangle \qquad \text{(since each } f(e) \text{ is nonnegative)} \quad \square$$

Maximum-Flow and Minimum-Cut Problems

Corollary 12.1.6 (Weak Duality) Let *f** be a maximum flow in an *s-t* network *N*, and let *K** be a minimum *s-t* cut in *N.* Then

$$val(f^*) \leq cap(K^*)$$

Proof: This follows immediately from proposition 12.1.5.

Corollary 12.1.7 (Certificate of Optimality) Let *f* be a flow in an *s-t* network *N* and *K* an *s-t* cut, and suppose that *val(f) = cap(K).*
Then flow *f* is a maximum flow in network *N*, and cut *K* is a minimum cut.

Proof: Let *f'* be any feasible flow in network *N.*
Proposition 12.1.5 and the premise give

$$val(f') \leq cap(K) = val(f)$$

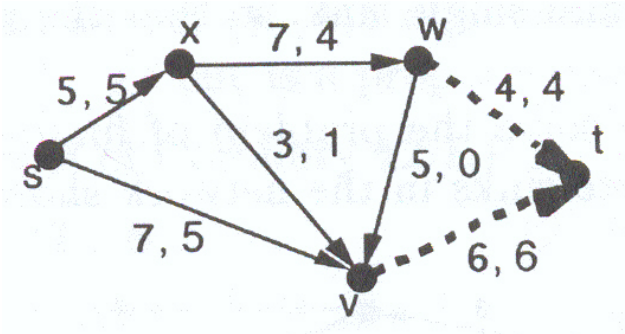
On the other hand, let $\langle V_s, V_t \rangle$ be any *s-t* cut. Proposition 12.1.5:

$$cap(K) = val(f) \leq cap\langle V_s, V_t \rangle$$

Therefore, *K* is a minimum cut. □

Example

Example The flow for the example network shown in the figure has value 10, which is also the capacity of the *s-t* cut $\langle\{s,x,v,w\},\{t\}\rangle$.
By corollary 12.1.7, both the flow and the cut are optimal for their respective problem.



A maximum flow and minimum cut.

Corollary 12.1.8 Let $\langle V_s, V_t\rangle$ be an *s-t* cut in a network *N*, and suppose that *f* is a flow such that

$$f(e)=\begin{cases} cap(e) & \text{if } e\in\langle V_s,V_t\rangle \\ 0 & \text{if } e\in\langle V_t,V_s\rangle \end{cases}$$

Then *f* is a maximum flow in *N,* and $\langle V_s,V_t\rangle$ is a minimum cut.

Solving the Maximum-Flow Problem

We will present an algorithm that originated by Ford and Fulkerson (1962).
Idea: increase the flow in a network iteratively until it cannot be increased any further → **augmenting flow path**.

Suppose that *f* is a flow in a capacitated *s-t* network *N,* and suppose that there exists a directed *s-t* path

$$P = \langle s, e_1, v_1, e_2, \ldots, e_k, t \rangle$$

in *N*, such that $f(e_i) < cap(e_i)$ for $i=1, \ldots, k$.

Then considering arc capacities only, the flow on each arc e_i can be increased by as much as $cap(e_i) - f(e_i)$.

But to maintain the conservation-of-flow property at each of the vertices v_i, the increases on all of the arcs of path *P* must be equal.

Thus, if Δ_P denotes this increase, then the largest possible value for Δ_P is $min\{cap(e_i\} - f(e_i)\}$.

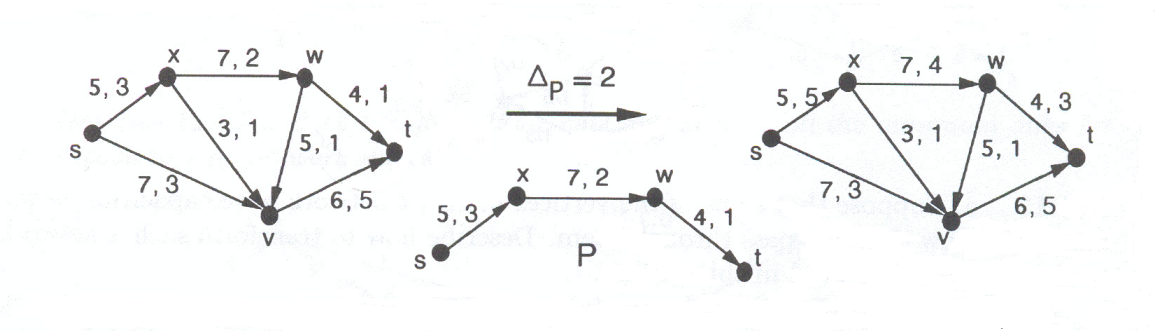
Solving the Maximum-Flow Problem

Example: Left side: the value of the current flow is 6.

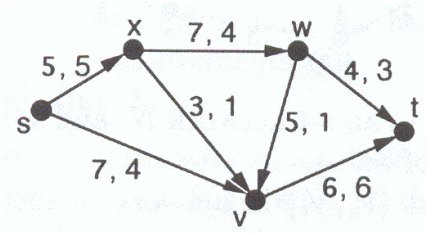
Consider the directed *s-t* path $P = \langle s,x,w,t \rangle$.

The flow on each arc of path P can be increased by $\Delta_P = 2$.

The resulting flow, which has value 8, is shown on the right side.



Using the directed path $\langle s,v,t \rangle$, the flow can be increased to 9. The resulting flow is shown right.

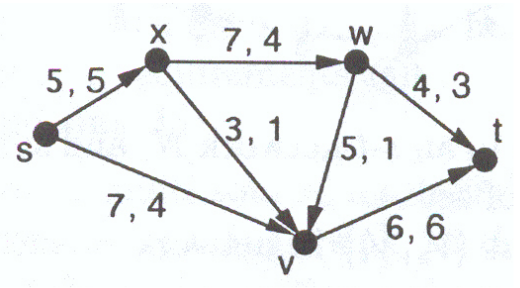


At this point, the flow cannot be increased any further along **directed *s-t paths***, because each such path must either use the arc directed from *s* to *x* or from *v* to *t*. Both arcs have flow at capacity.

Solving the Maximum-Flow Problem

However, the flow can be increased further.

E.g. increase the flow on the arc from source *s* to vertex *v* by one unit, decrease the flow on the arc from *w* to *v* by one unit, and increase the flow on the arc from *w* to *t* by one unit.



f-Augmenting Paths

Definition: An *s-t* **quasi-path** in a network *N* is an alternating sequence

$$\langle s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t \rangle$$

of vertices and arcs that forms an *s-t* path in the underlying undirected graph of *N.*

Terminology For a given *s-t* quasi-path

$$Q = \langle s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t \rangle$$

arc e_i is called a **forward arc** if it is directed from vertex v_{i-1} to vertex v_i and
arc e_i is called a **backward arc** if it is directed from v_i to v_{i-1}.

Clearly, a directed *s-t* path is a quasi-path whose arcs are all forward.

Example. On the *s-t* quasi-path shown below, arcs *a* and *b* are backward, and the three other arcs are forward.

f-Augmenting Paths

Definition: Let *f* be a flow in an *s-t* network *N.* An ***f*-augmenting path** Q is an *s-t* quasi path in *N* such that the flow on each forward arc can be increased, and the flow on each backward arc can be decreased.

Thus, for each arc *e* on an *f*-augmenting path Q,

$f(e) < cap(e)$,	if *e* is a forward arc
$f(e) > 0$	if *e* is a backward arc.

Notation For each arc *e* on a given *f*-augmenting path Q, let Δ_e be the quantity given by

$$\Delta_e = \begin{cases} cap(e) - f(e), & \text{if } e \text{ is a forward arc} \\ f(e), & \text{if } e \text{ is a backward arc} \end{cases}$$

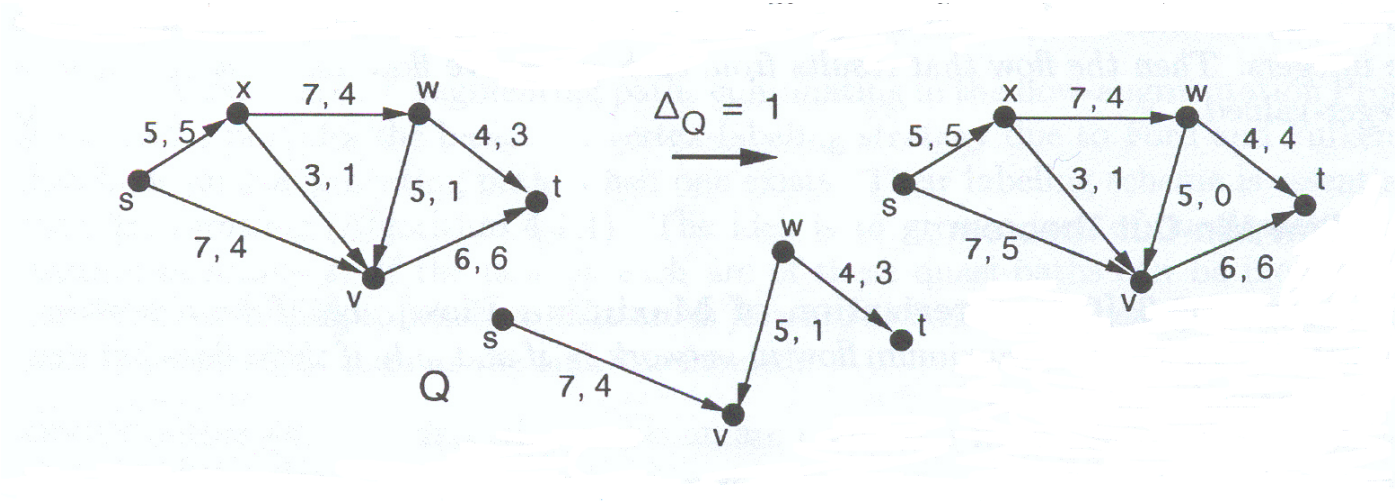
Terminology The quantity Δ_e is called the **slack on arc** *e.* Its value on a forward arc is the largest possible increase in the flow, and on a backward arc, the largest possible decrease in the flow, disregarding conservation of flow.

f-Augmenting Paths

Remark Conservation of flow requires that the change in the flow on the arcs of an augmenting flow path be of equal magnitude.
Thus, the maximum allowable change in the flow on an arc of quasipath Q is Δ_Q, where

$$\Delta_Q = \min_{e \in Q} \{\Delta_e\}$$

Example For the example network shown below, the current flow *f* has value 9, and the quasi-path Q = ⟨*s,v,w,t*⟩ is an *f*-augmenting path with $\Delta_Q = 1$.



flow augmentation

Proposition 12.2.1 (Flow Augmentation) Let *f* be a flow in a network *N*, and let Q be an *f*-augmenting path with minimum slack Δ_Q on its arcs.
Then the augmented flow *f'* given by

$$f'(e) = \begin{cases} f(e) + \Delta_Q, & \text{if } e \text{ is a forward arc of Q} \\ f(e) - \Delta_Q, & \text{if } e \text{ is a backward arc of Q} \\ f(e) & \text{otherwise} \end{cases}$$

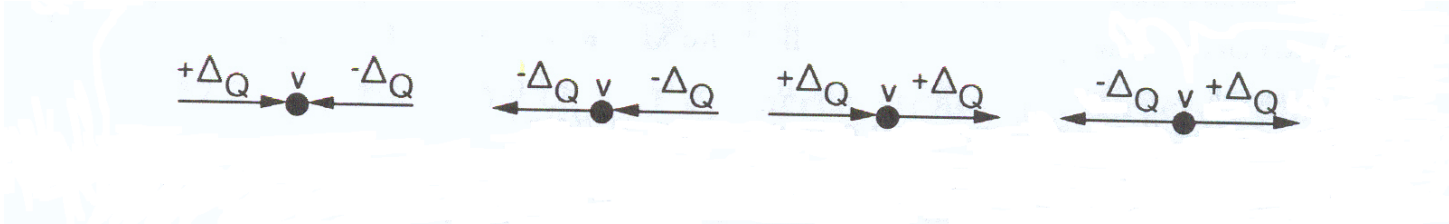
is a feasible flow in network *N* and *val(f')* = *val(f)* + Δ_Q.

Proof. Clearly, 0 ≤ *f'(e)* ≤ *cap(e)*, by the definition of Δ_Q.

The only vertices through which the net flow may have changed are those vertices on the augmenting path Q. Thus, to verify that *f'* satisfies conservation of flow, only the internal vertices of Q need to be checked.

f-Augmenting Paths

For a given vertex *v* on augmenting path Q, the two arcs of Q that are incident on *v* are configured in one of four ways, as shown below. In each case, the net flow into or out of vertex *v* does not change, thereby preserving the conservation-of-flow property.



It remains to be shown that the flow has increased by Δ_Q.

The only arc incident on the source *s* whose flow has changed is the first arc e_1 of augmenting path Q.

If e_1 is a forward arc, then $f'(e_1) = f(e_1) + \Delta_Q$, and

if e_1 is a backward arc, then $f'(e_1) = f(e_1) - \Delta_Q$. In either case,

$$val(f') = \sum_{e \in Out(s)} f'(e) - \sum_{e \in In(s)} f'(e) = \Delta_Q + val(f) \quad \square$$

Max-Flow Min-Cut

Theorem 12.2.3 [Characterization of Maximum Flow]

Let *f* be a flow in a network *N.*

Then *f* is a maximum flow in network *N* if and only if there does not exist an *f*-augmenting path in *N.*

Proof: Necessity (⇒) Suppose that *f* is a maximum flow in network *N*.

Then by Proposition 12.2.1, there is no *f*-augmenting path.

Proposition 12.2.1 (Flow Augmentation) Let *f* be a flow in a network *N*, and let Q be an *f*-augmenting path with minimum slack ΔQ on its arcs. Then the augmented flow *f'* given by

$$f'(e) = \begin{cases} f(e) + \Delta_Q, & \text{if } e \text{ is a forward arc of Q} \\ f(e) - \Delta_Q, & \text{if } e \text{ is a backward arc of Q} \\ f(e) & \text{otherwise} \end{cases}$$

is a feasible flow in network *N* and *val(f')* = *val(f)* + ΔQ.

→ assuming an *f*-augmenting path existed, we could construct a flow *f'* with *val(f')* > *val(f)* contradicting the maximality of *f.*

Max-Flow Min-Cut

Sufficiency ($\Leftarrow$) Suppose that there does not exist an *f*-augmenting path in network *N.*

Consider the collection of all quasi-paths in network *N* that begin with source *s*, and have the following property: each forward arc on the quasi-path has positive slack, and each backward arc on the quasi-path has positive flow.

Let V_s be the union of the vertex-sets of these quasi-paths.

Since there is no *f*-augmenting path, it follows that sink $t \notin V_s$.

Let $V_t = V_N - V_s$.

Then $\langle V_s, V_t \rangle$ is an *s*-*t* cut of network *N*. Moreover, by definition of the sets V_s and V_t ,

$$f(e) = \begin{cases} cap(e) & \text{if } e \in \langle V_s, V_t \rangle \\ 0 & \text{if } e \in \langle V_t, V_s \rangle \end{cases}$$

(if the flow along these edges *e* were not cap(e) or 0, these edges would belong to V_s!)

Hence, *f* is a maximum flow, by Corollary 12.1.8. □

Max-Flow Min-Cut

Theorem 12.2.4 [Max-Flow Min-Cut] For a given network, the value of a maximum flow is equal to the capacity of a minimum cut.

Proof: The *s-t* cut $\langle V_s, V_t \rangle$ that we just constructed in the proof of Theorem 12.2.3 (direction $\Leftarrow$) has capacity equal to the maximum flow. □

The outline of an algorithm for maximizing the flow in a network emerges from Proposition 12.2.1 and Theorem 12.2.3.

Algorithm 12.2.1: Outline for Maximum Flow

Input: an *s-t* network N.
Output: a maximum flow f^* in network N.

[*Initialization*]
For each arc e in network N
  $f^*(e) := 0$
[*Flow Augmentation*]
While there exists an f^*-augmenting path in network N
  Find an f^*-augmenting path Q.
  Let $\Delta_Q = \min_{e \in Q}\{\Delta_e\}$.
  For each arc e of augmenting path Q
    If e is a forward arc
      $f^*(e) := f^*(e) + \Delta_Q$
    Else (e is a backward arc)
      $f^*(e) := f^*(e) - \Delta_Q$
Return flow f^*.

Finding an *f*-Augmenting Path

The discussion of *f*-augmenting paths culminating in the flow-augmenting Proposition 12.2.1 provides the basis of a vertex-labeling strategy due to Ford and Fulkerson that finds an *f*-augmenting path, when one exists.

Their labelling scheme is essentially basic tree-growing.

The idea is to grow a tree of quasi-paths, each starting at source *s*.

If the flow on each arc of these quasi-paths can be increased or decreased, according to whether that arc is **forward** or **backward**, then an *f*-augmenting path is obtained as soon as the sink *t* is labelled.

Finding an *f*-Augmenting Path

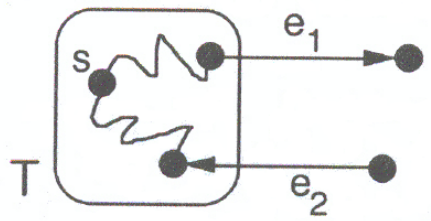
A **frontier arc** is an arc *e* directed from a **labeled** endpoint *v* to an **unlabeled** endpoint *w*.

For constructing an *f*-augmenting path, the frontier path *e* is allowed to be backward (directed from vertex *w* to vertex *v*), and it can be added to the tree as long as it has slack $\Delta_e > 0$.

Finding an *f*-Augmenting Path

Terminology: At any stage during tree-growing for constructing an *f*-augmenting path, let *e* be a frontier arc of tree *T*, with endpoints *v* and *w.*
The arc *e* is said to be **usable** if, for the current flow *f*, either

e is directed from vertex *v* to vertex *w* and $f(e) < cap(e)$, or
e is directed from vertex *w* to vertex *v* and $f(e) > 0$.



Frontier arcs e_1 and e_2 are usable if
$f(e_1) < cap(e_1)$ and $f(e_2) > 0$

Remark From this vertex-labeling scheme, any of the existing *f*-augmenting paths could result. But the efficiency of Algorithm 12.2.1 is based on being able to find „good“ augmenting paths.
If the arc capacities are irrational numbers, then an algorithm using the Ford&Fulkerson labeling scheme might not terminate (strictly speaking, it would not be an algorithm).

Finding an *f*-Augmenting Path

Even when flows and capacities are restricted to be integers, problems concerning efficiency still exist.

E.g., if each flow augmentation were to increase the flow by only one unit, then the number of augmentations required for maximization would equal the capacity of a minimum cut.

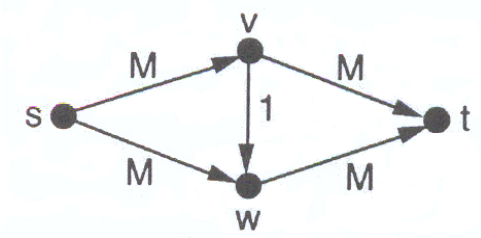
Such an algorithm would depend on the size of the arc capacities instead of on the size of the network.

Finding an *f*-Augmenting Path

Example: For the network shown below, the arc from vertex *v* to vertex *w* has flow capacity 1, while the other arcs have capacity *M*, which could be made arbitrarily large.

If the choice of the augmenting flow path at each iteration were to alternate between the directed path $\langle s,v,w,t \rangle$ and the quasi path $\langle s,w,v,t \rangle$, then the flow would increase by only one unit at each iteration.

Thus, it could take as many as $2M$ iterations to obtain the maximum flow.



Finding an *f*-Augmenting Path

Edmonds and Karp avoid these problems with this algorithm.

It uses **breadth-first search** to find an *f*-augmenting path with the smallest number of arcs.

Algorithm 12.2.2: Finding an Augmenting Path

Input: a flow f in an s-t network N.
Output: an f-augmenting path Q or a minimum s-t cut with capacity $val(f)$.

```
Initialize vertex set V_s := {s}.
Write label 0 on vertex s.
Initialize label counter i := 1
While vertex set V_s does not contain sink t
    If there are usable arcs
        Let e be a usable arc whose labeled endpoint v
         has the smallest possible label.
        Let w be the unlabeled endpoint of arc e.
        Set backpoint(w) := v.
        Write label i on vertex w.
        V_s := V_s ∪ {w}
        i := i + 1
    Else
        Return s-t cut ⟨V_s, V_N − V_s⟩.
Reconstruct the f-augmenting path Q by following backpointers,
 starting from sink t.
Return f-augmenting path Q.
```

FFEK algorithm: Ford, Fulkerson, Edmonds, and Karp

Algorithm 12.2.3 combines Algorithms 12.2.1 and 12.2.2

Algorithm 12.2.3: FFEK - Maximum Flow

Input: an s-t network N.
Output: a maximum flow f^* in network N.

[*Initialization*]
For each arc e in network N
  $f^*(e) := 0$
[*Flow Augmentation*]
Repeat
  Apply Algorithm 12.2.2 to find an f^*-augmenting path Q.
  Let $\Delta_Q = \min_{e \in Q}\{\Delta_e\}$.
  For each arc e of augmenting path Q
    If e is a forward arc
      $f^*(e) := f^*(e) + \Delta_Q$
    Else (e is a backward arc)
      $f^*(e) := f^*(e) - \Delta_Q$
Until an f^*-augmenting path cannot be found in network N.
Return flow f^*.

FFEK algorithm: Ford, Fulkerson, Edmonds, and Karp

Example: the figures illustrate algorithm 12.2.3.

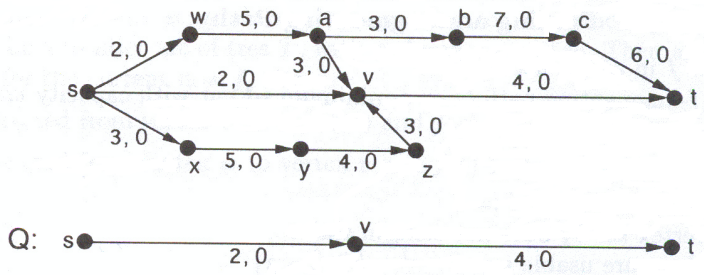


Figure 12.2.8 Iteration 0: $val(f) = 0$; augmenting path Q has $\Delta_Q = 2$.

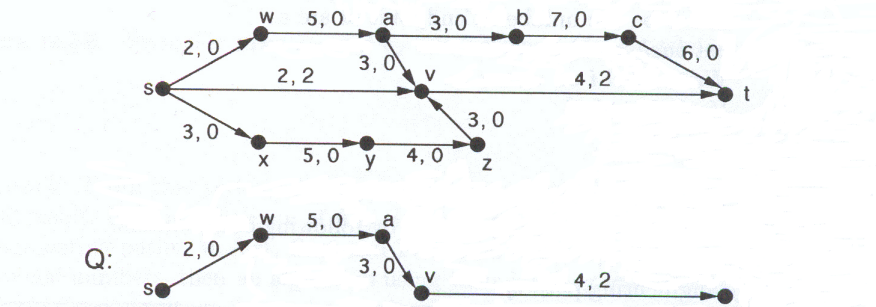


Figure 12.2.9 Iteration 1: $val(f) = 2$; augmenting path Q has $\Delta_Q = 2$.

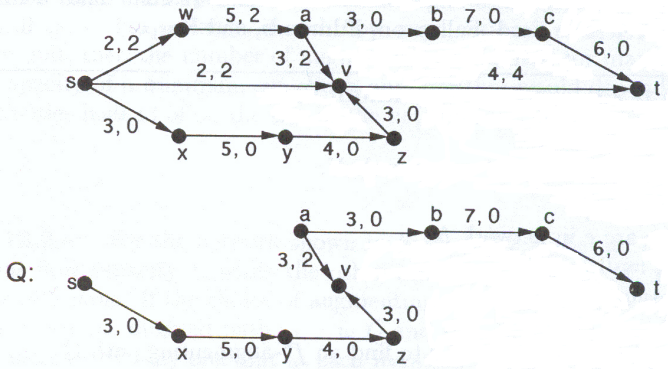
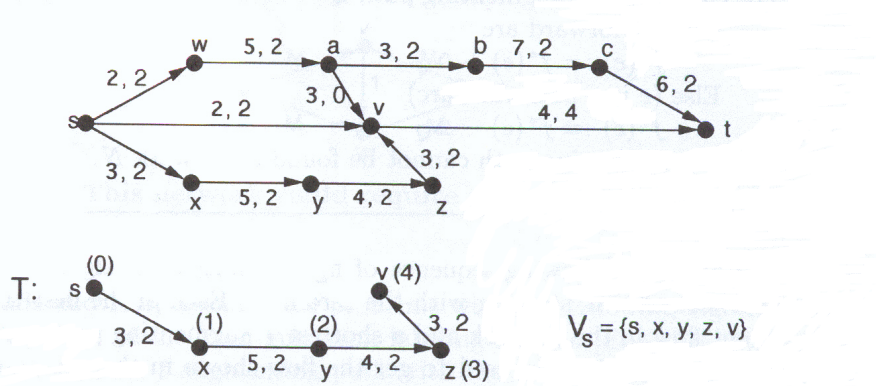


Figure 12.2.10 Iteration 2: $val(f) = 4$; augmenting path Q has $\Delta_Q = 2$.



e 12.2.11 Final iteration: $val(f) = 6 = cap\langle\{s, x, y, z, v\}, \{w, a, b, c, t\}\rangle$.

<{s, x, y, z, v}, {w, a, b, c, t}> is the *s-t* cut with capacity equal to the current flow, establishing optimality.

FFEK algorithm: Ford, Fulkerson, Edmonds, and Karp

At the end of the final iteration, the two arcs from source *s* to vertex *w* and the arc directed from vertex *v* to sink *t* form the minimum cut $\langle$ {*s,x,y,z,v* }, {*w,a,b,c,t*} $\rangle$. Neither of them is usable, i.e. the flow(e) = cap(e).

This illustrates the *s-t* cut that was constructed in the proof of theorem 12.2.3.