V2: Measures and Metrics (II)

- Betweenness Centrality

- Groups of Vertices

- Transitivity

- Reciprocity

- Signed Edges and Structural Balance

- Similarity

- Homophily and Assortative Mixing

Betweenness

Betweenness measures the extent to which a vertex lies on **paths between** other vertices.

The idea is usually attributed to Freeman (1977), but was already proposed in an unpublished technical report by Anthonisse a few years earlier.

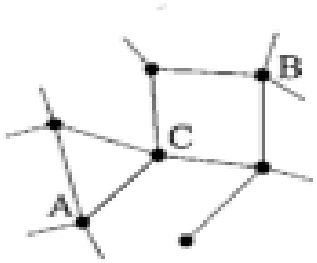
Vertices with high betweenness centrality may have considerable influence within a network e.g. by virtue of their control over information passing between others (think of data packages in a message-passing scenario).

The vertices with highest betweenness are the ones whose removal from the network will **disrupt** most communications between other vertices.

Betweenness Centrality

Let us assume an undirected network for simplicity.

Let n_{st}^{i} be 1 if vertex *i* lies on the geodesic path from *s* to *t*
and 0 if it does not or if there is no such path.



Vertices A and B are connected by 2 geodesic paths. Vertex C lies on both paths.

Then the **betweenness centrality** x_i is given by

$$x_i = \sum_{st} n_{st}^{i}$$

This definition counts separately the geodesic paths in either direction between each vertex pair.

The equation also includes paths from each vertex to itself.
Excluding those would not change the ranking of the vertices in terms of betweenness.

Also, we assume that vertices *s* and *t* belong to paths between *s* and *t.*

If there are two geodesic paths of the same length between 2 vertices, each path gets a weight equal to the inverse of the number of paths.

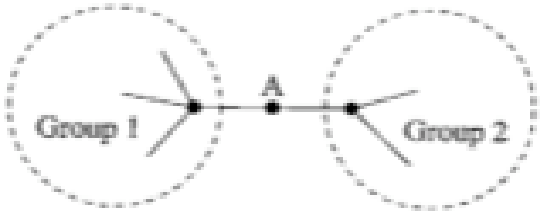
Betweenness Centrality

We may redefine n_{st}^{i} to be the number of geodesic paths from *s* to *t* that pass through vertex *i,* and define g_{st} to be the total number of geodesic paths from *s* to *t*.

Then, the betweenness centrality of x_i is

$$x_i = \sum_{st} \frac{n_{st}{}^{i}}{g_{st}}$$

where we adopt the convention that $n_{st}^{i} / g_{st} = 0$ if both n_{st}^{i} and g_{st} are zero.



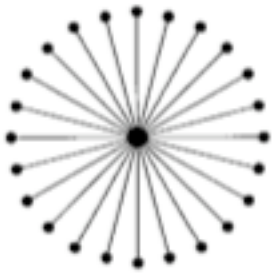
In this sketch of a network, vertex A lies on a bridge joining two groups of other vertices.
All paths between the groups must pass through A, so it has a high betweenness even though its degree is low.

Range of Betweenness Centrality

The betweenness values are typically distributed over a wide range.

The **maximum possible value** for the betweenness of a vertex occurs when the vertex lies on the shortest path between every other pair of vertices.

This occurs for the central vertex in a **star graph** that is attached to all other $n-1$ vertices by single edges.



A star graph.

The central vertex lies on all n^2 shortest paths between vertex pairs except for the $n-1$ paths from the peripheral vertices to themselves.

Thus, the betweenness centrality of the central vertex is $n^2 - n + 1$.

Range of Betweenness Centrality

In contrast, the **smallest possible value** of betweenness in a network with a single component is $2n - 1$,
since at a minimum each vertex lies on every path that starts or ends with itself.

There are $n - 1$ paths from others to the vertex,
$n - 1$ paths from a vertex to others and
one path from the vertex to itself.

In total, this gives $2n - 1$.

This situation occurs, for instance, when a network has a „leaf“ attached to it, which is a vertex connected to the rest of the network by just a single edge.

The ratio of largest and smallest possible betweenness values is thus

$$\frac{n^2 - n + 1}{2n - 1} \approx \frac{1}{2} n$$

For large networks, this range of values can become very large.

Betweenness Centrality for the film actor network

Taking again the example of the network of film actors,
the individual with the highest betweenness centrality
in the largest component of the actor network is

Spanish actor **Fernando Rey**

who played e.g. with Gene Hackman in
The French Connection.
Rey has a betweenness of 7.47×10^8.

The lowest score of any actor in the large component is just 8.91×10^5.
Thus, the betweenness values span a range of almost one thousand.

The second highest betweenness has Christopher Lee with 6.46×10^8.

Groups of Vertices

Many networks divide naturally into **groups** or **communities**.

In lectures V3 and V4, we will discuss some sophisticated computer methods that have been developed for detecting groups, such as hierarchical clustering and spectral clustering.

Today, we will start with cliques, plexes, cores, and components.

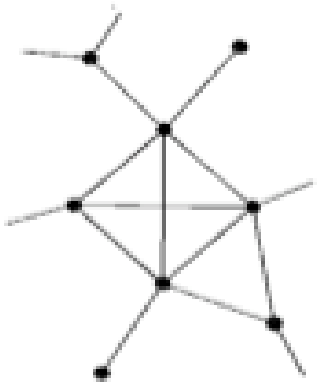
A **clique** is a subset of the vertices in an undirected network such that every member of the set is connected by an edge to every other member.

Newman defines cliques as maximal cliques so that there is no other vertex in the network that can be added to the subset while preserving the property that every vertex is connected to every other.

Other scientists distinguish between cliques and maximal cliques.

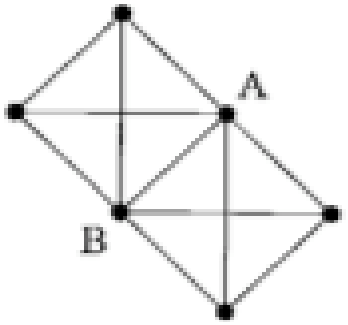
Cliques

A clique of 4 vertices within a network.



Cliques can also overlap.

In this network, vertices A and B
belong to 2 cliques of 4 vertices.



In a social network, a clique may be formed
e.g. by the co-workers in an office or
a group of classmates in school.

k-plex

Often, many circles of friends form only near-cliques rather than perfect cliques.

One way to relax the stringent requirement that every member is connected to every other member is the k-plex.

A **k-plex** of size *n* is a maximal subset of *n* vertices within a network such that each vertex is connected to at least $n - k$ of the others.

A 1-plex is the same as a clique.

In a 2-plex, each vertex must be connected to all or all-but-one of the others.

And so forth.

One can also specify that each member should be connected to a certain fraction (75% or 50%) of the other vertices.

k-core

The k-core is another concept that is closely related to the k-plex.

A **k-core** is a maximal subset of vertices such that each is connected to at least *k* others in the subset.

Obviously, a k-core of *n* vertices is also an (*n* – *k)*-plex.

However, the set of all *k*-cores for a given value of *k* is not the same as the set of all *k*-plexes for any value of *k*, since *n*, the size of the group, can vary from one *k*-core to another.

Also, unlike *k*-plexes and cliques, *k*-cores cannot overlap.

The *k*-core is of particular interest in network analysis for the practical reason that it is very easy to find the set of all *k*-cores in a network.

k-core: simple algorithm

A simple algorithm is to start with the whole network and remove from it any vertices that have degree less than *k*.

Such vertices cannot under any circumstances be members of a *k*-core.

In doing so, one will normally also reduce the degrees of some other vertices in the network that were connected to the vertices just removed.

So we go through the network again to see if there are any more vertices that now have degree less than *k* and, if there are, we remove those too.

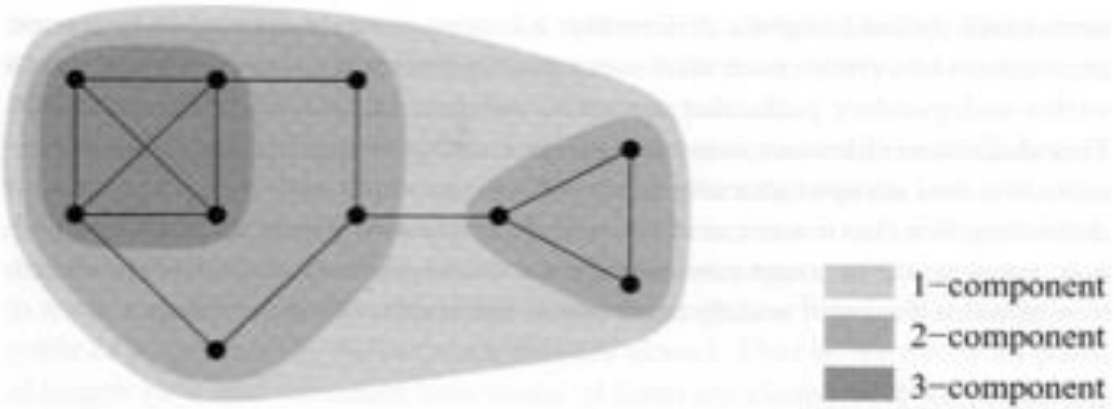
We repeatedly prune the network until no vertices remain with degree < *k*.

What is left, is by definition a *k*-core or a set of *k*–cores.

Components and k-components

A **component** in an undirected network is a maximal subset of vertices such that each is reachable by some path from each of the others.

A **k-component** (sometimes also called k-connected component) is a maximal subset of vertices such that each is reachable from each of the others by at least *k* vertex-independent paths.



The idea of *k*-components is connected with the idea of network robustness.

A pair of vertices connected by 2 independent paths cannot be disconnected by the failure of a single router.

Transitivity

A property very important in social networks, and useful to a lesser degree in other networks too, is **transitivity**.

If the „connected by an edge“ relation were transitive, it would mean that if vertex *u* is connected to vertex *v,* and *v* is connected to *w,* then *u* is also connected to *w*.

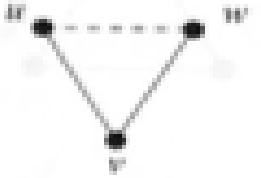
Perfect transitivity only occurs in networks where each component is a fully connected subgraph or clique. Perfect transitivity is therefore pretty much a useless concept for understanding networks.

However, **partial transitivity** can be very useful.

In many networks, particularly social networks, the fact that *u* knows *v,* and *v* knows *w,* doesn‘t guarantee that *u* knows *w* but makes it much more likely.

Transitivity

We quantify the level of transitivity in a network as follows:
if *u* knows *v,* and *v* knows *w,* then we have a path *uvw* of two edges in the network.
If *u* also knows *w,* we say that the path is **closed**.
It forms a loop of length 3, or a triangle.



We define the clustering coefficient to be the fraction of paths of length 2 in the network that are closed.

$$C = \frac{\text{number of closed paths of length 2}}{\text{number of paths of length 2}}$$

$C \in [0,1]$

C = 1 implies perfect transitivity; C = 0 implies no closed triads (happens e.g. for a tree topology or a squared lattice).

Transitivity

Social networks tend to have quite high values of the clustering coefficient.

E.g. the network of film actor collaborations has C = 0.20

A network of collaborations between biologists has C = 0.09

A network of who sends email to whom in a large university has C = 0.16.

These are typical values of social networks.

Reciprocity

The clustering coefficient measures the frequency with which loops of length 3 – triangles – appear in a network.

A triangle is the shortest loop in an undirected graph.

However, in a directed network, we can also have shorter loops of length 2.

What is the frequency of occurrence of such loops?

This is measured by the **reciprocity** what tells us how likely it is that a vertex that you point to also points to you.

E.g. on the World Wide Web if my web page links to your web page, how likely is it, on average, that yours link back again to mine?

Reciprocity

The **reciprocity** *r* is defined as the fraction of edges that are reciprocated.

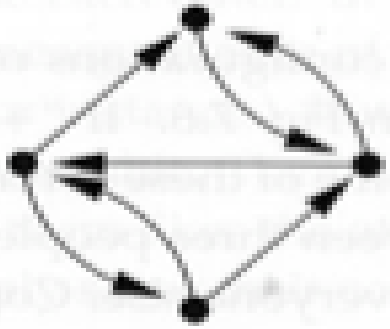
The product of adjacency matrix elements $A_{ij} A_{ji}$ is 1 if and only if there is an edge from *i* to *j* and an edge from *j* to *i* and is zero otherwise.

Thus, we can sum over all vertex pairs *i,j* to get an expression for the reciprocity

$$r = \frac{1}{m}\sum_{ij} A_{ij}A_{ji} = \frac{1}{m}\mathrm{Tr}\mathbf{A}^2$$

where *m* is, as usual,
the total number of directed edges in the network.
(„Tr“ stands for „trace“ = sum of diagonal elements of a matrix.)

In the right network with 7 directed edges,
4 are reciprocated, thus $r = 4/7 \approx 0.57$

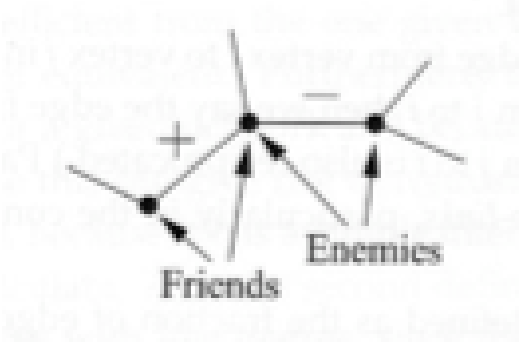


Also the WWW has $r = 0.57$

Signed edges and structural balance

In some social networks, and occasionally in other networks, edges are allowed to be either „positive“ or „negative“.

E.g. in an acquaintance network we could denote friendship by a positive edge and animosity by a negative edge.



Such networks are called **signed networks** and their edges **signed edges**.

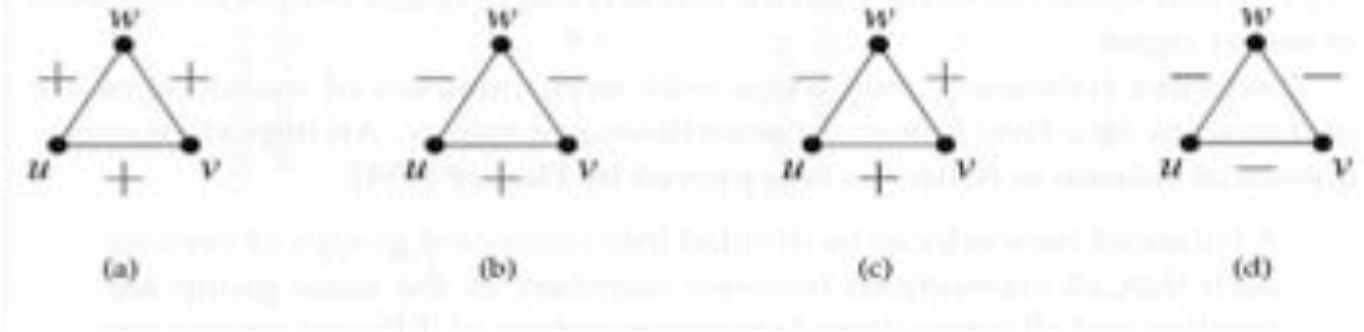
A negative edge is not the same as the absence of an edge.

A negative edge indicates, for example, 2 people who interact regularly but dislike each other.

The absence of an edge represents 2 people who do not interact.

Signed edges and structural balance

3 edges in a triangle in a signed network have the following possible configurations:



Configurations (a) and (b) are balanced and hence relatively stable.

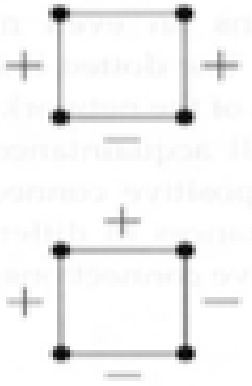
Configurations (c) and (d) are unbalanced and liable to break apart when interpreted as an acquaintance network.

We all know the „rule“ of „the enemy of my enemy is my friend“.
In (d), this rule is not obeyed.
Likely these 3 enemies will simply break up and go separate ways.

Signed edges and structural balance

The feature that distinguishes the 2 stable configurations from the unstable ones is that they have an even number of minus signs around the loop.

One can enumerate similar configurations for longer loops with e.g. *n = 4.*

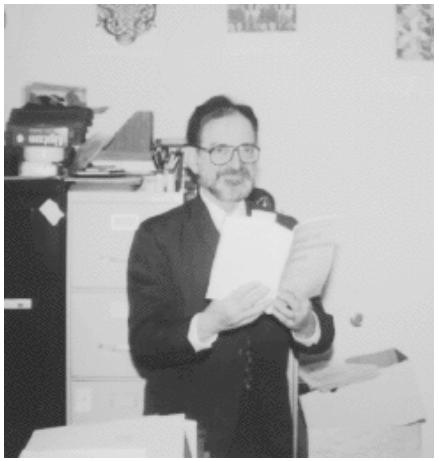


Surveys found that social networks contain far less unstable configurations than stable configurations with even numbers of minus signs.

Networks containing only loops with even numbers of minus signs are said to show **structural balance** or are simply termed **balanced**.

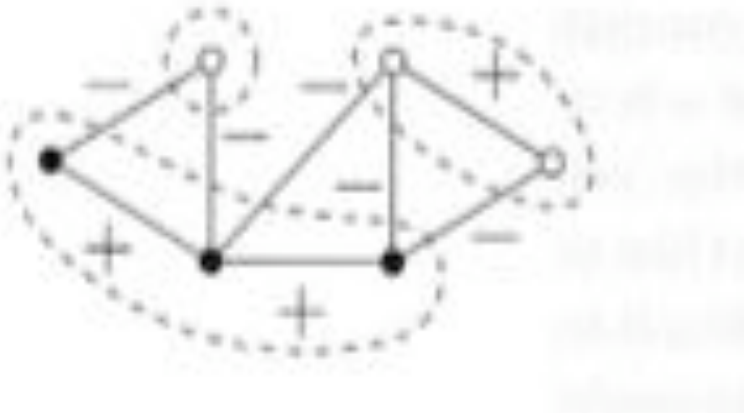
Structural balance

Frank Harary (see fig. right) proved:



A balanced network can be divided into connected groups of vertices such that all connections between members of the same group are positive and all connections between members of different groups are negative.

Shown on the right is a balanced, clusterable network.
Every loop in this network contains an even number of minus signs.



The dotted lines indicate the division of the network into **clusters** such that all acquaintances within clusters have positive connections and all acquaintances in different clusters have negative connections

Networks that can be divided into groups like this are termed **clusterable**.

http://www.cs.nmsu.edu/fnh/

Proof of Harary's theorem

Let us start by considering connected networks (they have one component).

We start with an arbitrary vertex and color it in one of two colors.

Then we color the other vertices according to the following algorithm:

1. A vertex *v* connected by a positive edge to another *u* that has already been colored gets the same color as *u.*

2. A vertex *v* connected by a negative edge to another *u* that has already been colored gets colored in the opposite color from *u*.

For most networks we may come upon a vertex whose color has already been assigned.
Then, a conflict may arise between this already assigned color and the new color that we would like to assign to it.

Proof of Harary's theorem

We will now show that this conflict can only arise if the network as a whole is unbalanced.

If while coloring a network we arrive at a already colored vertex, there must be another path by which this vertex can be reached from our starting point.

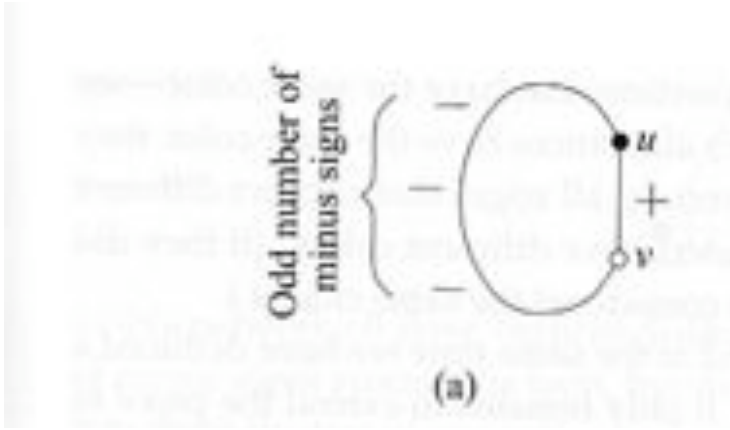
→ there is at least one loop in the network to which this vertex belongs.

If the network is balanced, every loop to which our vertex belongs must have an even number of negative edges around it.

Let us suppose that the color already assigned to the vertex is in conflict with the one we would like to assign it now.

There are 2 ways in which this could happen.

Proof of Harary's theorem

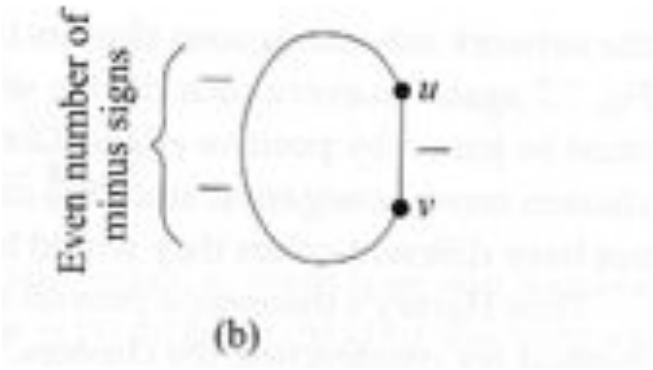


In case (a), vertex *u* is colored black and we move on to its neighbor *v* that is connected by a positive edge but already colored white.

If *u* and *v* have opposite colors, then around any loop containing them both there must be an **odd** number of minus signs, so that the color changes an odd number of times.

If there is an odd number of minus signs, the network is not balanced.

Proof of Harary's theorem



In the other case (b) vertices *u* and *v* have the same color but the edge between them is negative.

If *u* and *v* have the same color then there must be an even number of minus signs around the rest of the loop connecting them.

Together with the negative edge between *u* and *v* this gives again an odd total number of negative edges around the entire loop.

Hence, the network is again not balanced.

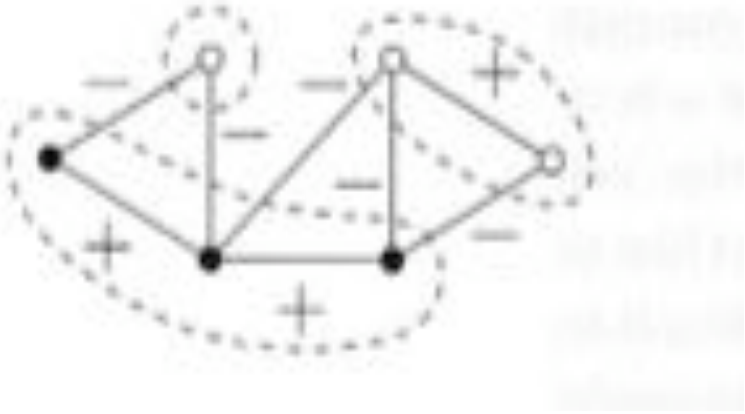
Proof of Harary's theorem

In either way, if we ever encounter a conflict about what color a vertex should have, then the network must be unbalanced.

Turned around, balanced networks will not lead to such conflicts so that we can color the entire network with just two colors obeying the rules.

When this is completed, we simply divide the network into contiguous clusters of vertices that have the same color.

In every cluster, since all vertices have the same color, they must be joined by positive edges.



Conversely, all edges that connect different clusters must be negative since the clusters have different colors. This concludes the proof.

Proof of Harary's theorem

Our proof of Harary's theorem also led to a method for constructing the clusters.

The proof can be easily extended to networks with more to one component because we could simply repeat the proof for each component separately.

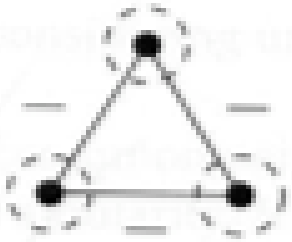
The practical importance of Harary's theorem is based on the observation that many real social networks are found naturally to be in a balanced or mostly balanced state.

Is the inverse of this theorem also true?

Are clusterable networks necessarily balanced?

Clusterable networks

No.



In this network, all 3 vertices dislike each other, so there is an odd number of minus signs around the loop.

But there is no problem dividing the network into 3 clusters of one vertex each such that everyone dislikes the members of the other clusters.

This network is clusterable but not balanced.

Similarity

Another central concept in social networks is that of **similarity** between vertices.

E.g. commercial dating services try to match people with others based on presumed similarity of their interests, backgrounds, likes and dislikes.

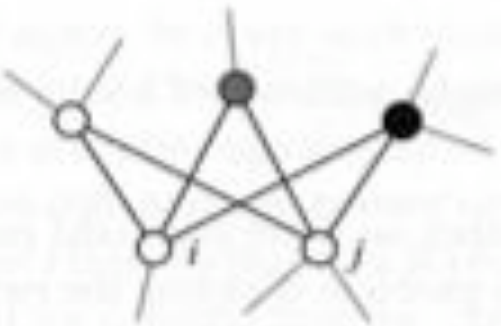
For this, one would likely use attributes of the vertices.

Here, we will restrict ourselves to determining similarity between the vertices of a network using the information contained in the network structure.

There are two fundamental approaches for constructing meaures of network similarity, called **structural equivalance** and **regular equivalence**.

Structural equivalence

Two vertices in a network are **structurally equivalent** if they share many of the same network neighbors.

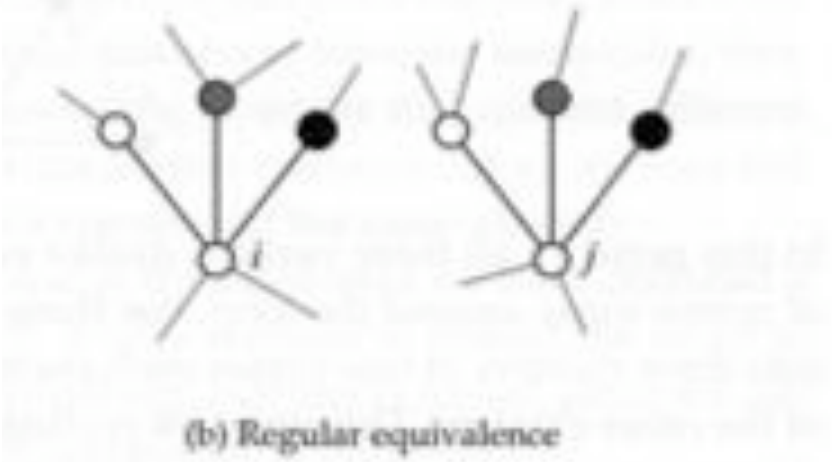


(a) Structural equivalence

In this figure, the two vertices *i* and *j* share 3 common neighbors (two black vertices and one white vertex).
Both also have other neighbors that are not shared.

Regular equivalence

2 regularly equivalent vertices do not necessarily share the same neighbors, but they have neighbors who are themselves similar.



(b) Regular equivalence

E.g. two history students at different universities may not have any friends in common, but they can still be similar in the sense that they both know a lot of other history students, history instructors, and so forth.

We will focus here on mathematical measures to quantify structural equivalence because these measures are considerably better developed than those for regular equivalence.

Cosine similarity

We will now introduce measures of structural equivalence and concentrate on undirected networks.

The simplest and most obvious measure of structural equivalence is a count of common neighbors n_{ij} of 2 vertices *i* and *j*:

$$n_{ij} = \sum_k A_{ik} A_{kj}$$

However, this number is difficult interpret.

If 2 vertices have 3 common neighbors, is that a lot or a little?

→ we need some form of normalization.

One strategy would be to divide this by the total number of vertices *n* in the network, because this is the maximal possible number of common neighbors.

Cosine similarity

However, this would penalize vertices with low degree.

A better measure would allow for the varying degree of vertices.

Such a measure is the **cosine similarity**.

In geometry, the inner or dot product of 2 vectors **x** and **y** is given by

$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}| \cos\theta$, where $|\mathbf{x}|$ is the magnitude of **x** and θ the angle between the 2 vectors.

Rearranging, we can write the cosine of the angle as

$$cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$$

Cosine similarity

Salton proposed that we regard the *i*-th and *j*-th rows (or columns) of the adjacency matrix as two vectors and use the cosine of the angle between them as similarity measure.

By noting that the dot product of two rows is $\sum_k A_{ik}A_{kj}$

this gives us the following similarity

$$\sigma_{ij} = \cos\theta = \frac{\sum_k A_{ik}A_{kj}}{\sqrt{\sum_k {A_{ik}}^2}\sqrt{\sum_k {A_{jk}}^2}}$$

Assuming our network is an unweighted simple graph, the elements of the adjacency matrix take on only the values 0 and 1, so that $A_{ij}^2 = A_{ij}$ for all *i,j.*

Cosine similarity

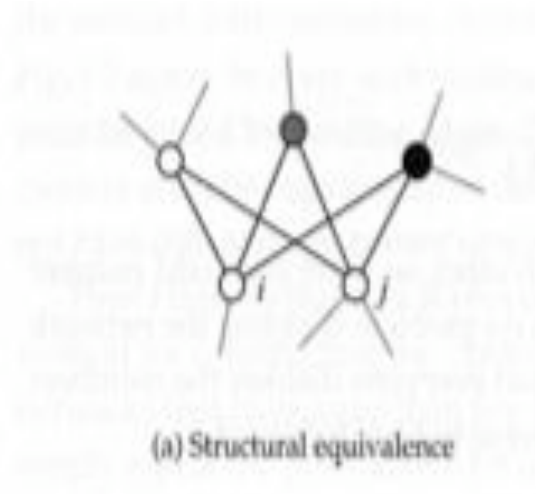
Then $\sum_k {A_{ik}}^2 = \sum_k A_{ik} = k_i$ where k_i is the degree of vertex *i*. Thus

$$\rho_{ij} = \frac{\sum_k A_{ik}A_{kj}}{\sqrt{k_i k_j}} = \frac{n_{ij}}{\sqrt{k_i k_j}}$$

The cosine similarity of *i* and *j* is therefore the number of common neighbors n_{ij} of the two vertices divided by the geometric mean of their degrees.

In this example, the cosine similarity of *i* and *j* is $\sigma_{ij} = \frac{3}{\sqrt{4\times5}} = 0.671$

If one or both vertices have degree zero, we set $\sigma_{ij} = 0$.



(a) Structural equivalence

Pearson coefficients

An alternative way to normalize the count of common neighbors is to compare it to the expected value when vertices choose their neighbors at random.

Suppose vertices *i* and *j* have degrees k_i and k_j , respectively.

How many neighbors should we expect them to have?

Imagine that vertex *i* chooses k_i neighbors uniformly at random from *n* possible ones (*n* -1 if self-loops are not allowed).

In the same manner, vertex *j* chooses k_j random neighbors.

Pearson coefficients

For the first neighbor that j chooses, there is a probability of k_i / n that j chooses a neighbor of i.

The same probability applies to the next choices.
(We neglect the small probability of choosing the same neighbor twice.)

→ the expected number of common neighbors is $k_i k_j / n$

A reasonable measure of similarity between two vertices is the actual number of common neighbors minus the expected number they would have if they chose their neighbors at random.

Pearson coefficients

$$\sum_k A_{ik}A_{jk} - \frac{k_i k_j}{n} = \sum_k A_{ik}A_{jk} - \frac{1}{n}\sum_k A_{ik}\sum_l A_{jl}$$

$$= \sum_k A_{ik}A_{jk} - n\langle A_i\rangle\langle A_j\rangle$$

$$= \sum_k \left[A_{ik}A_{jk} - \langle A_i\rangle\langle A_j\rangle\right]$$

$$= \sum_k (A_{ik} - \langle A_i\rangle)\left(A_{jk} - \langle A_j\rangle\right)$$

Here $\langle A_i \rangle$ denotes the mean $n^{-1}\sum_k A_{ik}$ of the elements of the *i* th row of the adjacency matrix.

This equation is *n* times the covariance $\mathrm{cov}(A_i, A_j)$ of the two rows of the adjacency matrix.

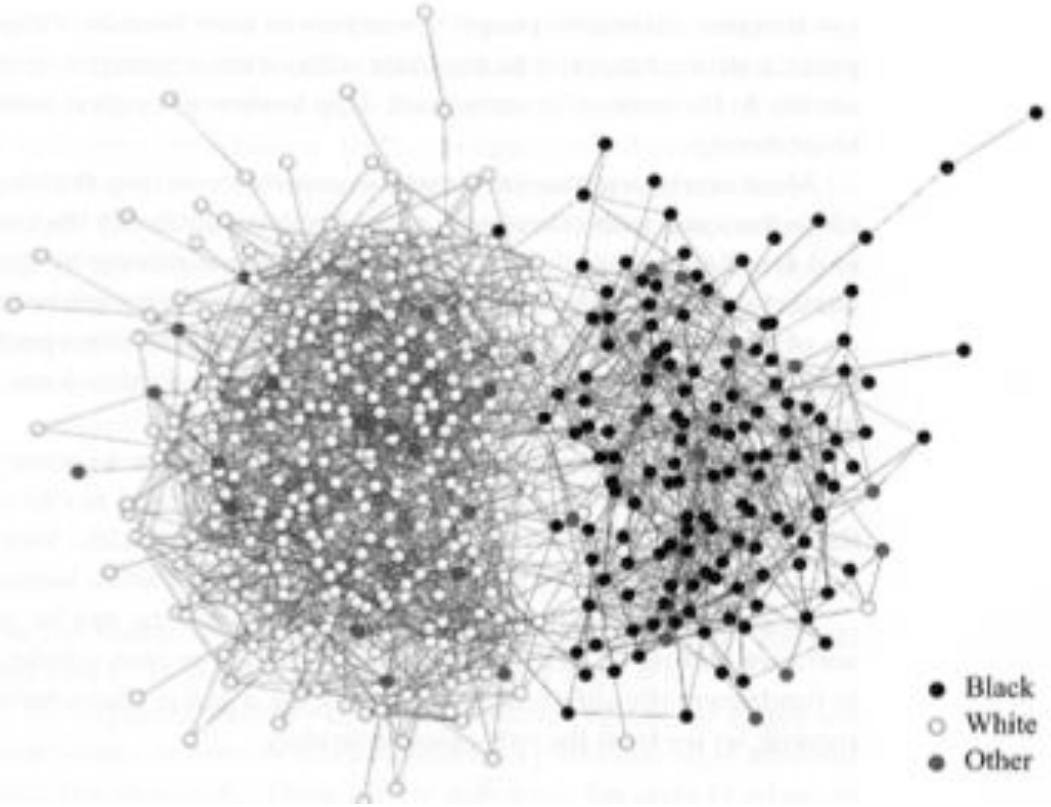
Pearson coefficients

Normalizing this as we did with the cosine similarity, so that it ranges between -1 and 1, gives the standard **Pearson correlation coefficient**

$$r_{ij} = \frac{cov(A_i, A_j)}{\sigma_i \sigma_j} = \frac{\sum_k (A_{ik} - \langle A_i \rangle)(A_{jk} - \langle A_j \rangle)}{\sqrt{\sum_k (A_{ik} - \langle A_i \rangle)^2} \sqrt{\sum_k (A_{jk} - \langle A_j \rangle)^2}}$$

Homophily or Assortative Mixing

Shown below is a friendship network among 470 students in a US high school (ages 14-18 years). The vertices are colored by race.



The figure shows that the students are sharply divided between a group mainly consisting of white children and one that contains mainly black children.

Homophily or Assortative Mixing

This observation is very common in social networks.

People have apparently a strong tendency to associate with others whom they perceive as being similar to themselves in some way.

This tendency is termed **homophily** or **assortative mixing**.

Assortative mixing is also seen in some nonsocial networks,
e.g. in citation networks where papers from one field cite papers from the same field.

How can one quantify assortative mixing?

Quantify assortative mixing

Find the fraction of edges that run between vertices of the same type and subtract from this the fraction of edges we would expect if edges were positioned at random without regard for vertex type.

c_i : class or type of vertex i , $c_i \in [1 \ldots n_c]$
n_c : total number of classes

The total number of edges between vertices of the same type is

$$\sum_{\text{edges } (i,j)} \delta(c_i, c_j) = \frac{1}{2} \sum_{ij} A_{ij} \delta(c_i, c_j)$$

Here $\delta(m,n)$ is the Kronecker delta (δ is 1 if $m = n$ and 0 otherwise). The factor ½ accounts for the fact that every vertex pair i,j is counted twice in the sum.

Quantify assortative mixing

Now we turn to the expected number of edges between vertices if the edges are placed randomly.

Consider a particular edge attached to vertex i which has degree k_i.

By definition, there are $2m$ ends of edges in the entire network where m is the total number of edges.

If connections are made randomly, the chances that the other end of our particular edge is one of the k_j ends attached to vertex j is thus $k_j / 2m$.

Counting all k_i edges attached to i , the total expected number of edges between vertices i and j is then $k_i \, k_j / 2m$

Quantify assortative mixing

The expected number of edges between all pairs of vertices of the same type is

$$\frac{1}{2}\sum_{ij}\frac{k_i k_j}{2m}\delta(c_i, c_j)$$

where the factor ½ avoids double-counting vertex pairs.

Taking the difference between the actual and expected number of edges gives

$$\frac{1}{2}\sum_{ij}A_{ij}\delta(c_i, c_j) - \frac{1}{2}\sum_{ij}\frac{k_i k_j}{2m}\delta(c_i, c_j) = \frac{1}{2}\sum_{ij}\left(A_{ij} - \frac{k_i k_j}{2m}\right)\delta(c_i, c_j)$$

Typically one does not calculate the number of such edges but the fraction, which is obtained by dividing this by *m*

$$Q = \frac{1}{2m}\sum_{ij}\left(A_{ij} - \frac{k_i k_j}{2m}\right)\delta(c_i, c_j)$$

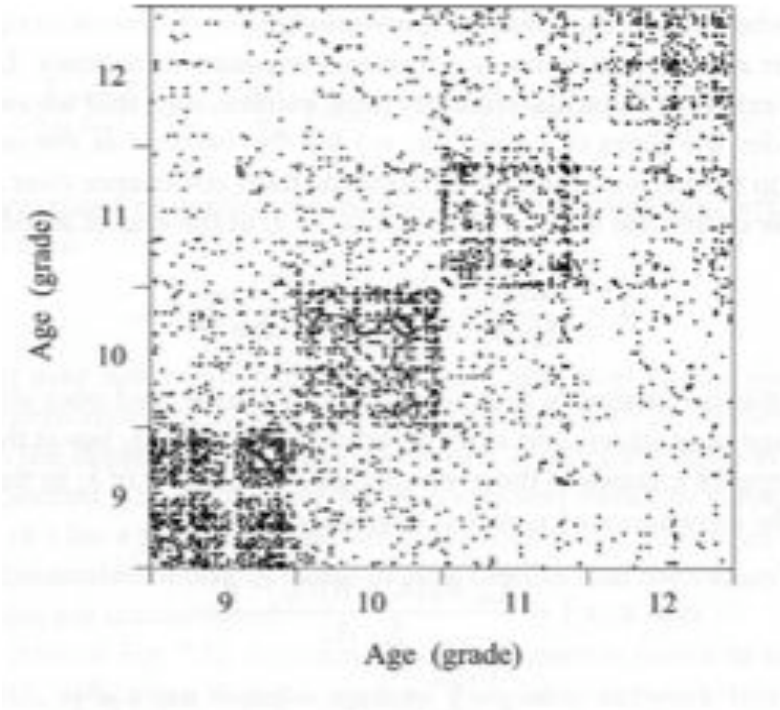
This quantity Q is called the **modularity**. In the student network Q=0.305.

Assortative mixing by scalar characteristics

When considering networks with associated scalar characteristics like age, income, or gene expression, we could study whether network vertices with similar values of this scalar characteristics tend to be connected together more often than those with different values.

This figure is for the same group of US schoolchildren, but now as a function of age. Each dot corresponds to a pair of friends sorted by their grades (school years).

Younger children form a very homogenous cluster in the bottom left corner, older children also have friends in the other grades.



Assortative mixing by scalar characteristics

A good way of quantifying this tendency is by the **covariance** (no derivation here):

$$cov(x_i, x_j) = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) x_i, x_j$$

where x_i, x_j now replaces the Kronecker symbol.